JOURNAL OF FINANCIAL DECISION MAKING
Volume 7, Number 2, December 2011

The Science and Art of Position Sizing

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Abstract

This paper will discuss position sizing based upon three different models: Portfolio theory, Kelly betting and the Alternative investment model which represent the 1% rule. We will then construct a simple trend following trading model where an investor is faced with a hefty 88% chances of a losing trade. This paper shows than an investor can overcome such obstacle by engaging in systematic risk management and by equalizing risk over changes in expected return. We have also shown that an investor's return is an increasing function of how long he stays in the market rather than how much risk he takes on.

Keywords: position size, trend following, risk management

1. Introduction and Literature Review

Financial markets tend to have certain characteristics that need to be understood in order to make better investment decisions. Conrad & Kaul (1988) found that expected return is financial markets tend to be positive serial correlated. Engle (1982) founds that volatility tend to be positive serial correlated. Given these characteristics how can we invest money in an optimal way? There are many competing models on how to investment money in an optimal way i.e. maximize expected return and minimize risk. Authors such as Covel (2004) is advocating the standard trend following approach which selects securities with large positive expected return and low return volatility. Markowitz (1959) takes the trend following model a bit further by looking at it from a portfolio perspective where cross correlation in return can reduce even more return noise than by trading individual securities. Sharpe (1964) takes the reasoning even further by suggesting that an investor should balance long and short positions to reduce market risk. There is however other prominent authors that are approaching things a little bit different. Kelly (1956) points out that in order for us to maximize expected return for a symmetrical return distribution our position size should be 2 * p(success) – 1. This is a very interesting finding. However sometime is can be difficult to accurately estimate your probability of success especially if you are a new investor that does not have a long history of performance data. What position size should such an investor have? Authors such as Monte and Swope (2008) explain that the best way for an investor to minimize risk is to apply the 1% rule. Such rule essentially equalizes risk over changes in expected return. Since changes in expected return are a very frequent empirical phenomenon such rule becomes highly important. Seykota (2001) further explains that an investor need to know what his total risk is i.e. if all my stops were triggered at the same time what would my maximum loss be? This paper will discuss and explore some of these concepts more in detail.

2. Modelling Framework

We will start our discussion by reviewing three somewhat competing theories when it comes to minimizing risk in financial markets. The first theory is the traditional portfolio model where the objective is to use trend following and cross correlation of returns to cancel out as much return noise as possible. For example we can assume that we have a portfolio of two assets. The covariance matrix for such portfolio is given by:

$$cov[portfolio] = \begin{bmatrix} w[1]^2 * var[1] & w[1] * w[2] * cov[1,2] \\ w[1] * w[2] * cov[1,2] & w[2]^2 * var[2] \end{bmatrix} \tag{1}$$

If we sum all the elements in such a covariance matrix we get

$$cov[portfolio] = w[1]^2 * var[1] + 2 * w[1] * w[2] * cov[1,2] + w[2]^2 * var[2] = 0 \tag{2}$$

We now assume that stocks 1's weight is given by w[1] = 1 which gives us:

$$cov[portfolio] = var[1] + 2 * w[2] * cov[1,2] + w[2]^2 * var[2] = 0 \tag{3}$$

We now want to minimize portfolio variance so we take the partial derivative with respect to w[2] and set it equal to zero which give us:

$$2 * cov[1, 2] + 2 * w[2] * var[2] = 0 \tag{4}$$

We now solve for w[2] which gives us:

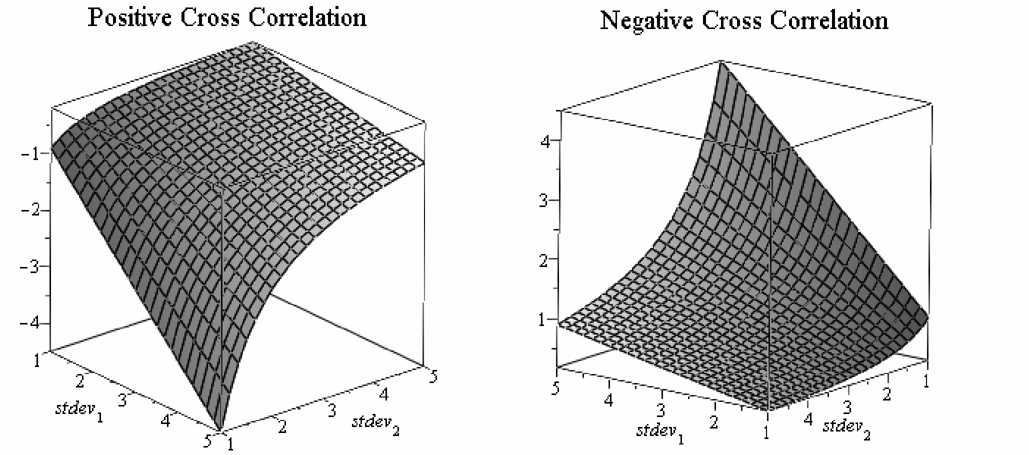
$$w[2] = -\frac{cov[1,2]}{var[2]} \tag{5}$$

Note that if we assume that we take a short position in stock1 w[1] = –1 instead of a long position w[1] = 1 then the only difference is that the above expression becomes positive. We should also now note that $cov[1,2] = stdev[1] * stdev[2] * corr[1,2]$ and that $var[2] = stdev[2]^2$ which means that we can write our previous equation as:

$$w[2] = -\frac{[stdev[1] * corr[1,2]]}{stdev[2]} \tag{6}$$

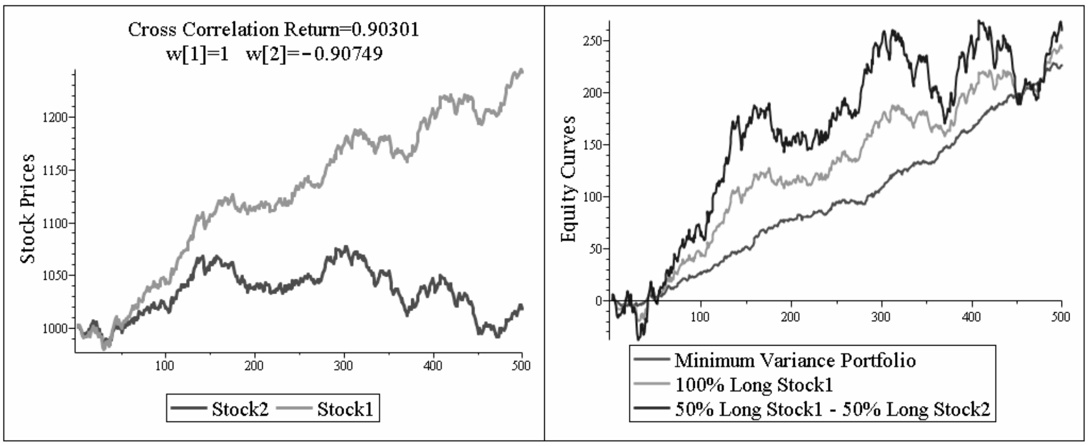
We can now plot such an equation as seen in exhibit-1 below. We can see that when the return of security1 and security2 are negative cross correlated then we will buy a large position in security2 when security2 has a low return variance. The larger the return variance becomes for security2 the smaller our position we will become. We can see that when the return of security1 and security2 are positive cross correlated then we will short a large position in security2 when security2 has a low return variance. The larger the return variance becomes for security2 the smaller our position we will become.

Exhibit-1 Position Sizing and Minimum Variance Ratio



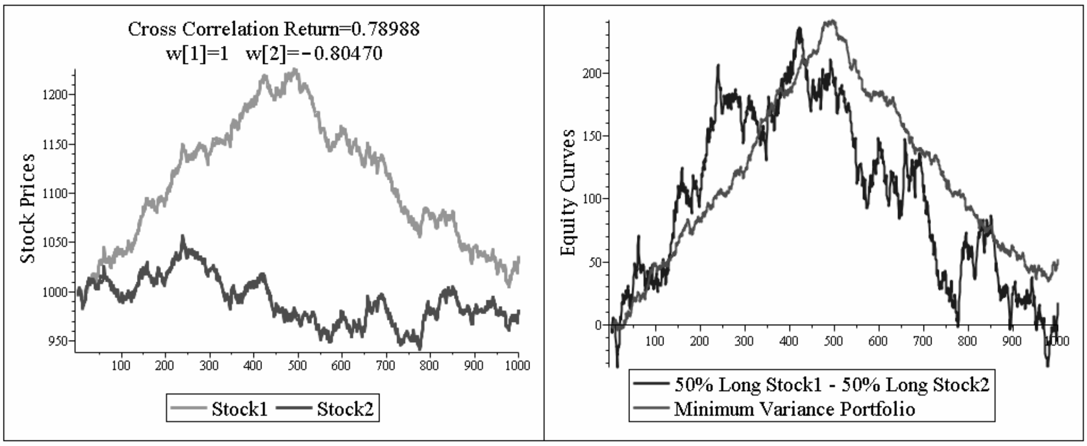
We can now run a simulation as seen in exhibit-2 on the next page. Stock1 is defined as the return generator since stock1 has a large positive expected return i.e. a large drift. Stock2 is defined as our hedge security. Stock2 has the same return volatility as stock 1 but have no drift i.e. a pure unit root. We can see that the equity curve for our minimum variance portfolio is much smoother than a 50-50 portfolio and 100 percent portfolio in stock1. This has to do with the fact that stock1 and stock2's returns are strongly positively crossed correlated so our large short position in stock2 will create negative cross correlated return hence reducing portfolio variance i.e. return noise. We can also see that since we have a significant short position in stock2 we are giving up return in order to get a more stable equity curve. We can see that both the 50-50 portfolio and the 100 percent portfolio in stock1 have larger return.

Exhibit-2 A Simulated Hedge Position in Stock2



The problem with the above example, which looks very nice in theory, is that we have assumed a constant expected return. In exhibit-3 below we can see an example with a non constant expected return. We can see that our hedged portfolio provide little protection.

Exhibit-3 Changes in Expected Return



An investor can solve the previous problem with either frequent rebalancing to maximize the Sharpe ratio or by applying the concepts we are going to discuss now. The second theory that we are going to discuss is Kelly betting. In the Kelly betting strategy expected return is the most important component in order to maximizing portfolio returns. Hence such model solves the previous model shortcoming. We assume that wealth in the next period $W[t+1]$ is a function of wealth in this period $W[t]W[t]$ plus the wealth that you gain or lose on your gamble $B * W[t] * R$ where B is the bet size (0..1) and R is the investment return (0..1). We can therefore write wealth in the next period as:

$$W[t+1] = W[t] + B * W[t] * R \tag{7}$$

We now subtract W[t] on both sides

$$W[t+1] - W[t] = B * W[t] * R \tag{8}$$

We now divide by W[t] to get the percentage return:

$$\frac{W[t+1]-W[t]}{W[t]} = B * R \tag{9}$$

The expected percentage return is therefore given by:

$$E\left[\frac{W[t+1]-W[t]}{W[t]}\right] = E[B*R] = B*E[R] \tag{10}$$

We now maximize the expected percentage return with respect to the bet size B

$$B^* = \frac{\partial E\left[\frac{W[t+1]-W[t]}{W[t]}\right]}{\partial B} = E[R] \tag{11}$$

This simply means that our optimal bet size should be proportional to our expected return in order to maximize expected percentage returns. For example if the minimum expected return is -0.3 and the maximum expected return is 0.3 then

$$B^* = \frac{ER}{0.3}$$

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We can also derive a more general expression for the optimal bet size in the Kelly model. Again wealth in the next period is given by: $W[t+1] = W[t] + B * W[t] * R$ which can be written as. We now divide both sides by:

$$\frac{W[t+1]}{W[t]} = (1 + B * R) \tag{12}$$

We now take the logarithm on both sides:

$$ln\left[\frac{W[t+1]}{W[t]}\right] = ln(1 + B * R) \tag{13}$$

We now note that the expression on the left-hand side corresponds to our log returns i.e.

$$W[t+1] = W[t] * \exp(r) \rightarrow \frac{W[t+1]}{W[t]} = \exp(r) \rightarrow ln\left[\frac{W[t+1]}{W[t]}\right] = r \tag{14}$$

We now take the expectations on both sides to get our expected return:

$$E\left[ln\left[\frac{W[t+1]}{W[t]}\right]\right] = E[ln(1 + B * R)] \tag{15}$$

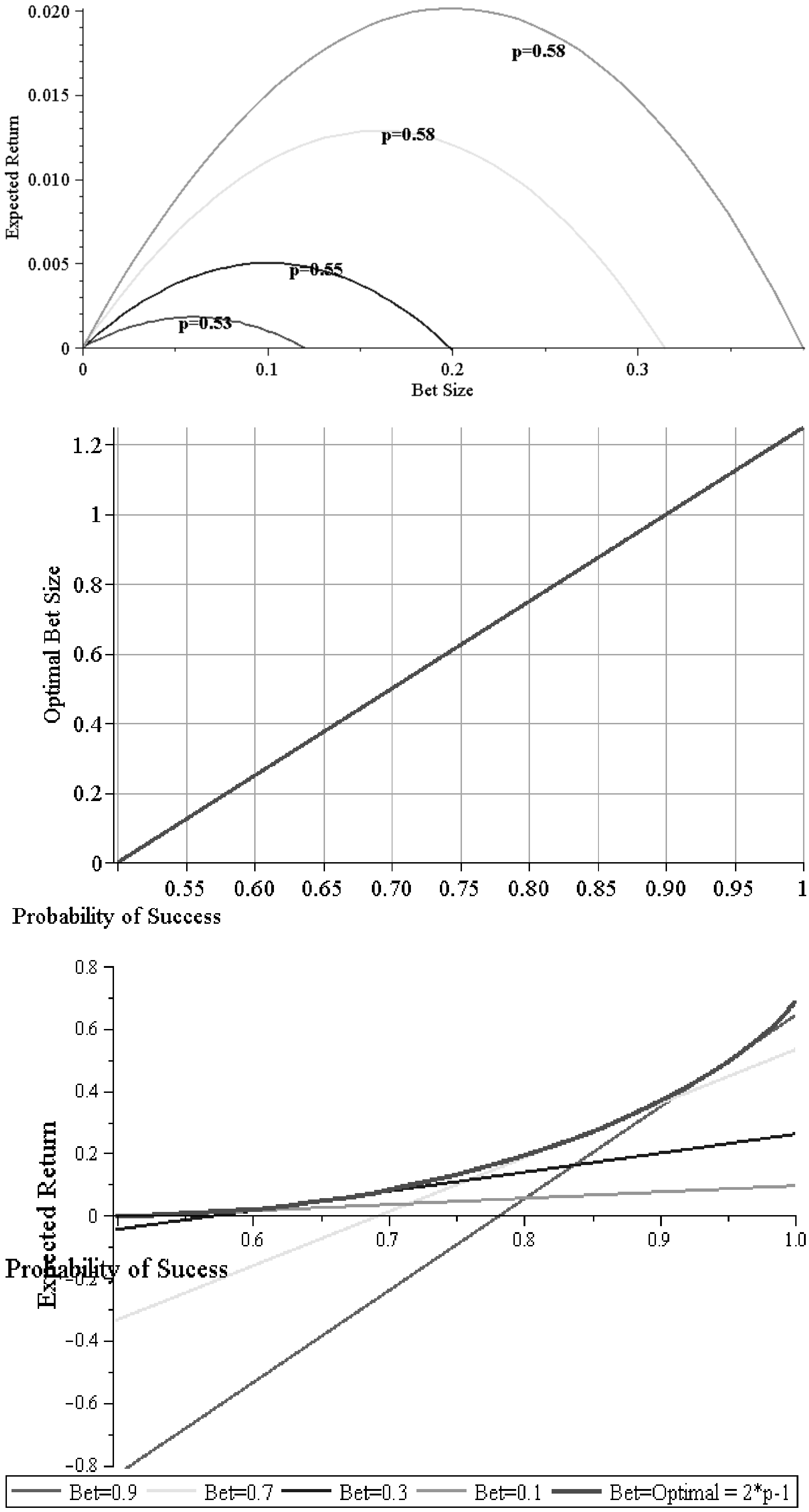
We now note that if we assume that we have a symmetrical return distribution where p is the probability of getting a return of +1 and $(1 - p)$ is the probability of getting a return of -1 then we can rewrite the above equation as follows:

$$E\left[ln\left[\frac{W[t+1]}{W[t]}\right]\right] = p * ln(1 + B * (+1)) + (1 - p) * ln(1 + B * (-1)) \tag{16}$$

Such an equation is plotted in the first figure in exhibit-4 on the next page. If we now take the partial derivative of our previous equation with respect to the bet size B and set it equal to zero and solve for B we will get the expression for the optimal bet size which turns out to be $B^* = 2 * p - 1$ $B^* = 2 * p - 1$. Such an equation is plotted in the second figure in exhibit-4. If we now plug in the expression for our optimal bet size in the previous equation we get the last expression as seen below. Such an equation is plotted in the third figure in exhibit-4.

$$E\left[ln\left[\frac{W[t+1]}{W[t]}\right]\right] = p * ln(2p) + (1 - p) * ln(2 - 2p) \tag{17}$$

Exhibit-4 Kelly Betting and Symmetric Return Distributions



The previous example can also easily be extended to include asymmetric return distributions as well. Now instead of +1 and –1 returns we can represent a positive return with *x* and a negative return with *y* which means that our previous equation becomes:

$$E\left[ln\left[\frac{W[t+1]}{W[t]}\right]\right] = p*ln(1+B*(+x))+(1-p)*ln(1+B*(-y)) \tag{18}$$

We again take the partial derivative with respect to the bet size *B* and set it equal to zero which gives us the following equation:

$$\frac{p*x}{1+B*x}-\frac{(1-p)*y}{1-B*y}=0 \tag{19}$$

We now solve for B to get the expression for the optimal bet size which turns out to be:

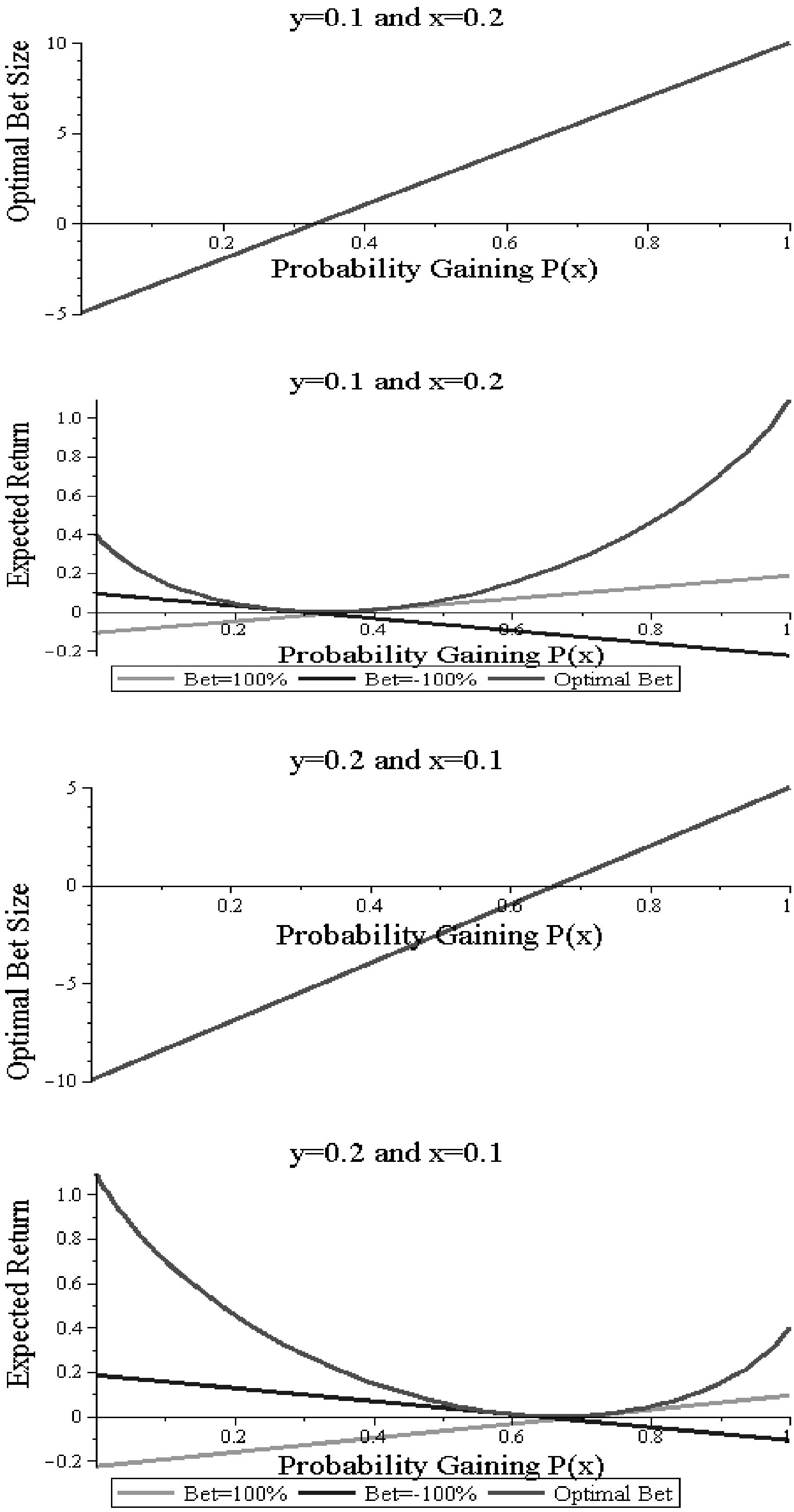
$$B* = \frac{p*x-y+y*p}{y*x} \tag{20}$$

If we plug in such an expression into the previous equation we get the expression for the expected return for a given optimal bet size:

$$p*ln\left(1+\frac{p*x-y+y*p}{y}\right)+(1-p)*ln\left(1-\frac{p*x-y+y*p}{x}\right) \tag{21}$$

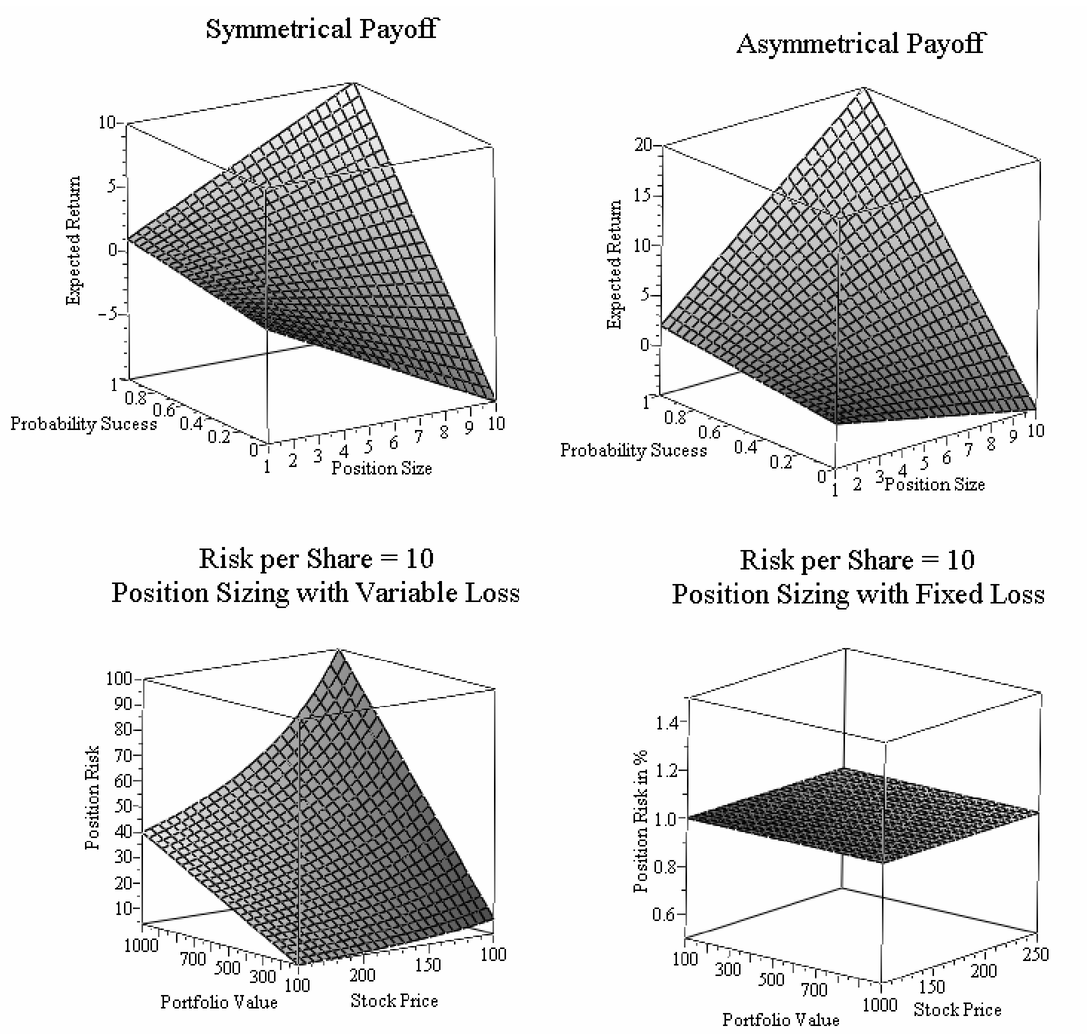
We can in exhibit-5 on the next page see the optimal bet size for different values of p where p is the probability of having a positive return. We can see that when x > y then even though p is low we still manage to achieve positive expected return. This has to do with the fact that the return distribution is asymmetrical. We can see that the optimal position size is negative up until p = 0.3 then the optimal position size turns positive. We can also see that the optimal bet size clearly outperforms the other strategies and that the expected return grows exponentially. Another thing to note is that the expected return is decreasing up until p = 0.3 and then starts to increase. For x < y the expected return is decreasing up until p = 0.7. We have illustrated the dynamics related to asymmetric Kelly betting further in appendix-1.

Exhibit-5 Kelly Betting and Asymmetric Return Distributions



Now from a trend following perspective where all the securities that we trade have large expected return and low return volatility i.e. trend and where the probability of success is low and in general is difficult to estimate. How do we approach position sizing then? This leads us to the third theory which I have chosen to call the alternative investment theory also known as the 1% rule. Such rule is very simple but also very effective because it equalizes risk over changes in expected return. We have in exhibit-6 below illustrated some of the dynamics. In the first two figures we can see how expected return changes when the position size is increased both for a symmetrical and asymmetrical payoff. The expected return for the symmetrical payoff example is given by prob*(+1*bet) + (1-prob)*(-1*bet) and the expected return for the asymmetrical payoff example is given by prob*(+2*bet) + (1-prob)*(-0.5*bet). The point with these two figures is to show that the difference in expected return, for different probability of success, is increasing when the position size is increasing.

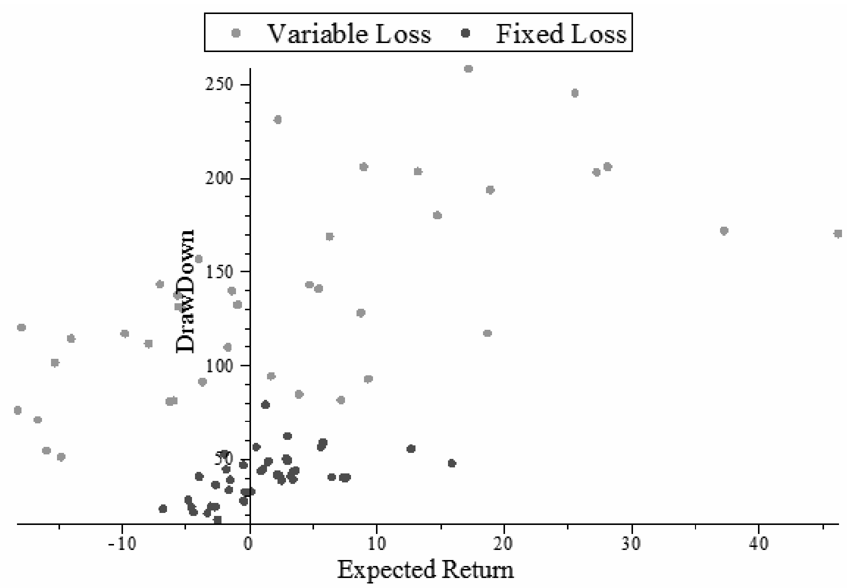
Exhibit-6 Position Sizing and Expected return



This is quite interesting. It is also important to note that the difference between expected return is smallest the closer we get to a probability of success equal to 0.5. However, at such point the expected return is zero so such point is not optimal. The figures illustrate that it is also highly important to look at differences in expected returns for different probability of success i.e. the more you bet the bigger such difference becomes hence risk is increasing. A point with minimal risk in theory would have two characteristics i) positive expected return ii) an expected return which is independent of the probability of success (a straight line on a saddle path). Such point would represent a stable equilibrium.

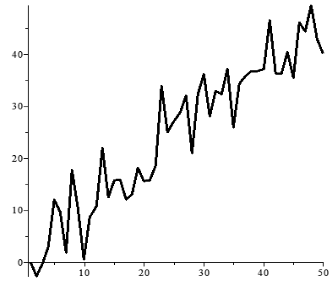
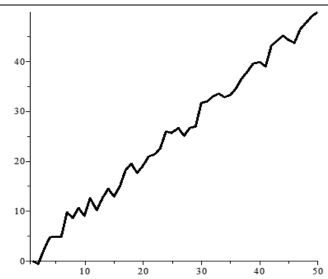
The third and fourth figures in exhibit-5 illustrate the benefit of the 1% rule i.e. the difference between a position size with fixed and variable loss. The risk per share is defined as the difference between the stock price and the trailing stop loss exit. In this example we have assume that such risk-per-share is 10. The bigger such a difference is the more risk per share we are taking on. In the variable loss example position risk is defined as risk-per-share*(portfolio value/stock price). So for example if our portfolio value is 1000 and the stock price is 250 then we can buy 4 stocks and hence our risk is 10*4 = 40. Alternatively if the portfolio value is 1000 and the stock price is 500 then we can buy 2 stocks and hence our risk is 10*2= 20. The conclusion is that the larger the stock price is the lower the risk becomes and vice versa the smaller the stock price is the more risk we are taking on. The problem with doing thing this way is that it introduces a significant amount of volatility. When we have a fixed loss position size we can see that the position size is independent of the portfolio value and the stock price which is a good thing. Position risk as a % of portfolio value is defined as 100*((Portfolio Value*0.01/ risk-per-share)*risk-per-share / Portfolio Value) = 0.01. The first expression Portfolio Value*0.01/ risk-per-share tells us how many stocks we should buy to make sure we only lose 1% of our portfolio. If we multiply such an expression with risk-per-share we get the total amount of risk. If we divide such an expression with the portfolio value we get the % risk of portfolio value. We can see in exhibit-7 below the difference between a trailing stop loss strategy with a variable loss position size and a trailing stop loss strategy with a fixed loss position size. We can see that variable loss position size has much higher volatility. In this example we have assumed that the expected return for the unit root has a stochastic element which is more realistic.

Exhibit-7 Variable vs. Fixed Position Sizing Drawdown



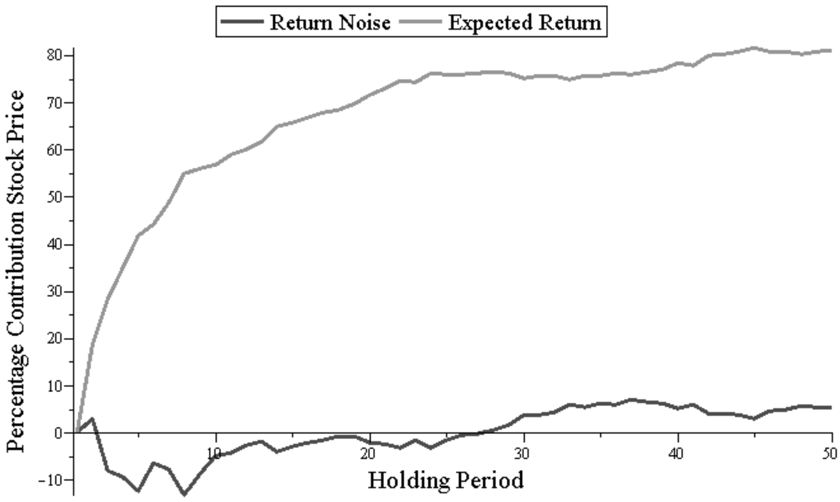
I have in exhibit-8 on the next page illustrated the difference between a position size with a fixed and variable loss more in detail. Again the risk per share is given by the entry price minus the stop loss exit price. For example if we assume that the entry price is 45.5 $ and the stop loss exit price is 43.50 $ then the risk per share or trailing distance is given by 45.5 - 43.5= 2 $. The position size according to the 1% of portfolio value rule is given by 0.01*value trading account/risk-per-share. For example if the trading account is 100 000 and the risk-per-share is 2 then the position size is (0.01*100 000)/2 = 500. Such rule makes sure that we don't lose more than 1% of our trading account on any single trade ie 500*2 = 1000 which is exactly 1% of 100 000.

Exhibit-8 Variable vs. Fixed Loss Position Size

Position Sizing with Variable Loss			
	Example-1	**Example-2**	
			
Portfolio Value (A)	1000	1000	
Stock Price	100	250	
Number of Shares Bought (B)	10	4	
Stop Price	90	240	
Risk per Share (C)	10	10	
Total Risk in % A (B*C/A)	0.1	0.04	
Position Sizing with Fixed Loss (The 1% of portfolio value rule)			
	Example-1	**Example-2**	
			
Portfolio Value (A)	1000	1000	
Stock Price	100	250	
Number of Shares Bought (B=A*0.01/C)	1	1	
Stop Price	90	240	
Risk per Share (C)	10	10	
Total Risk in % A (B*C/A)	0.01	0.01	

Now one reason why changes in expected return are important is because in the short run returns are driven by return noise but over time expected return corresponds to almost 100% of the stock price. This can be seen in exhibit-9 on the next page. Note that for a pure unit root the impact of expected return is zero since it has no drift. However, a pure unit root represents a very constrained model of reality due to its strict parametric requirements. We can see in the exhibit on the next page that the return noise is to a large extent canceled out and the only thing remain over time is expected return. This highlights the fact that managing expected return should become a number one priority for any investor.

Exhibit-9 Stock Price, Trends and Expected Returns



3. Theoretical Model of Trading Returns

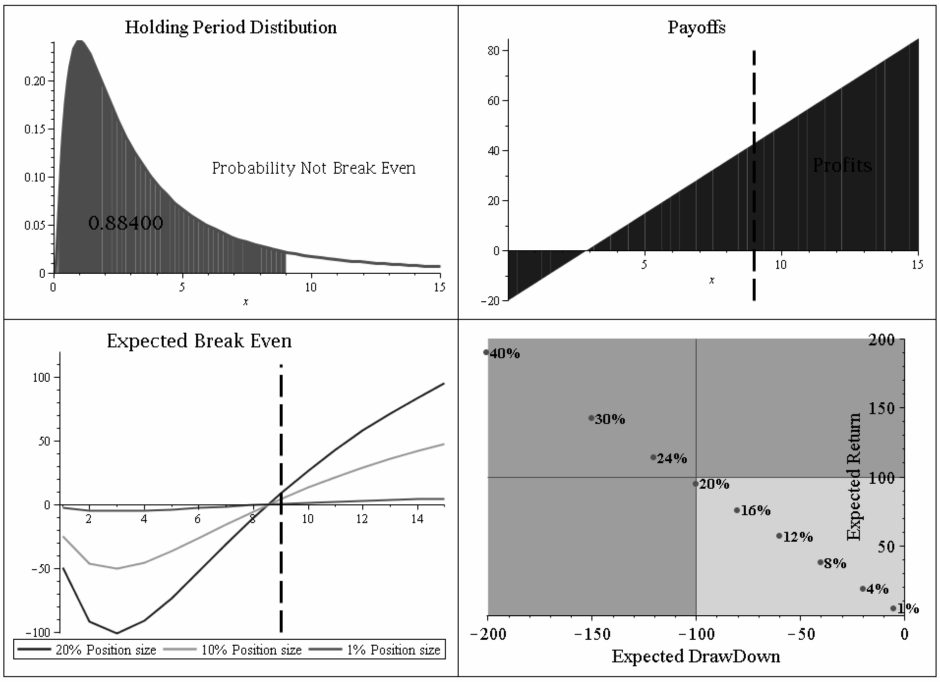
We will now discuss a simple model to highlight the challenges an investor is exposed to and the benefit of using such a simple position sizing rule as the 1% rule. Let assume that we have a trend following investor which buys securities with high risk adjusted returns and exit his positions with a trailing stop loss. In the first figure on the next page we can see the probability for different holding periods i.e. we can see that the majority of trades are exited before time period 3. The holding period distribution (pdf) is log normal with a mean of 1 and scale parameter of 1. In the second figure we can see the payoff for the different holding periods. We can see that all trades with a holding period less than three are negative trades. This has to do with the transaction cost. The expected return is given by the expression below where z is a position size parameter which takes a value between 0 and 1.

$$ER = \int pdf * f(x) * P\,dt \quad where \quad f(x) = -20 + 7 * x \quad P = z * 100 \tag{22}$$

$$pdf = \begin{Bmatrix} 0 & x < 0 \\ \dfrac{\sqrt{2} * \exp\left(-\dfrac{(\ln(x)-1)^2}{2}\right)}{2 * x * \sqrt{\pi}} & otherwise \end{Bmatrix} \tag{23}$$

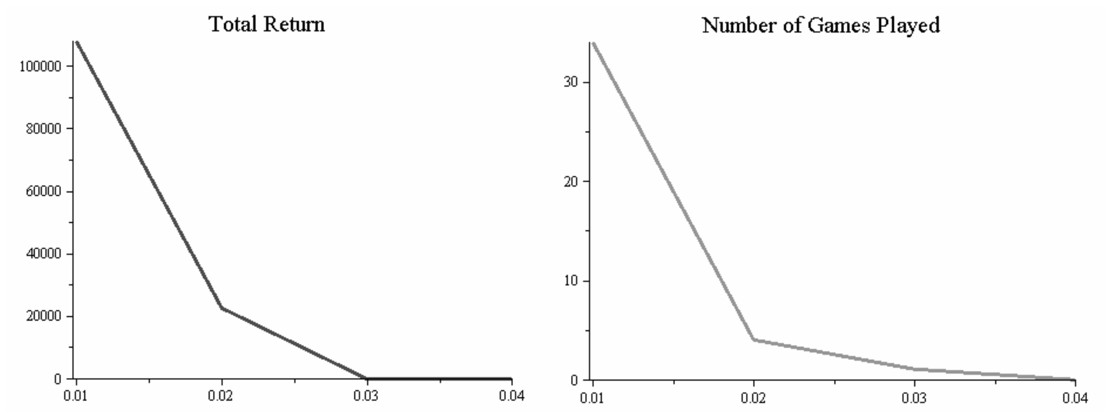
In the third figure we can see the expected breakeven which is 9. We can see that the expected breakeven is independent on the position size. The reason why the expected breakeven is 9 and not 3 has to do with the fact that the probability distribution is not uniformed. We can see that we have an 88% probability that our trade will be a losing trade i.e. will not even breakeven. In the fourth figure we have plotted the expected drawdown vs. expected return. Our portfolio value is assumed to 100 so all the different bet sizes that have an expected drawdown that is larger than -100 in the fourth figure are not realistic i.e. they do not represent valid position sizes. We can see that all bet sizes have a positive expected return and that expected return grows linearly with the position size i.e. the more you bet the more you can make. However, an investor's goal is not only to maximize expected return but also minimize expected drawdown.

Exhibit-10 A Simple Trend Following Trading Game



We now assume that every player i.e. player1%, player2%, player3% and player4% play one hundred games. Each player will be stopped out from the game if their cumulative return at time t is smaller than -100 i.e. the initial portfolio value. The returns are then added up from the starting point to the exit period for each player. If the player managed to play 100 games without going bankrupt i.e. the player play all 100 games and is not stopped out prematurely then such a player are allowed to play another round. If the player is stopped out prematurely because he went bankrupt i.e. if the return is smaller than -100 ie -250 etc then the player is not allowed to play anymore. Previously a player's return was a function of risk i.e. the larger the bet size was the more the player could potentially loss or win. What we now can see is that the amount of time that you are allowed to play the game is much more important than taking large risk. This is illustrated in exhibit-11. Player10% and Player20% are shooting for high returns but the problem is that they goes bankrupt before they even get a chance to play the game over a long period of time. Player1% plays the game for a much longer time than the other players and hence player1% is also the player that gets the highest total return. When an investor are faced with asymmetric return distributions with a large probability of failure i.e. a 0.88 probability of a losing trade then an investors ability to conserve capital and his ability to systematically take managed risk becomes crucial for his long term success.

Exhibit-11 Total Return and Trading Time



4. Concluding Remarks

We have in this paper discussed different ways that an investor can approach position sizing. We have created a simple trend following model where the investor faced an 88% chances of failure but still managed to extract wealth and maintain a smooth equity curve by using the 1% rule. Such a high probability of a losing trade might sound a lot but empirical evidence has shown that trend following investor's systematically loss more times than they win. Their return distributions are asymmetrical in sense that when they win they win much more than when they lose. The way they manage to achieve such a distribution is by using systematic risk management i.e. trailing stop losses and conservative position sizing. By placing small controlled bets in a wide range of trending markets (diversification) where all trades have the same risk levels i.e. the equalization of risk over changes in expected return they manage to control risk and maximize expected return over time.

It is also interesting to note that all financial markets by nature are asymmetric i.e. a stock can easily bubble or triple its value but the stock price cannot decrease with more than 100%. Also wins and losses are asymmetrical. You start with a capital of 200. In year one you lose 20% so your capital will be 160. In year two you will need a return of 25% (160*1.25 = 200) just too breakeven. The general trend following rule is that small frequent loses are preferred over large infrequent losses. Such rule sounds very simple in theory but observations have shown over and over again that people are very reluctant to take losses due to behavioral biases. The notion that a stock will come back and then the investor will get his lost money back is a very soothing and therapeutic game a lot distressed investor's play. They prefer to take a gamble that the stock will come back up rather than losing a fraction of their money. The empirical evidence however suggests that when a market are exposed to changes in expected return it will takes years before price get back to "normal". Position sizing and systematic risk management therefore becomes crucial for success.

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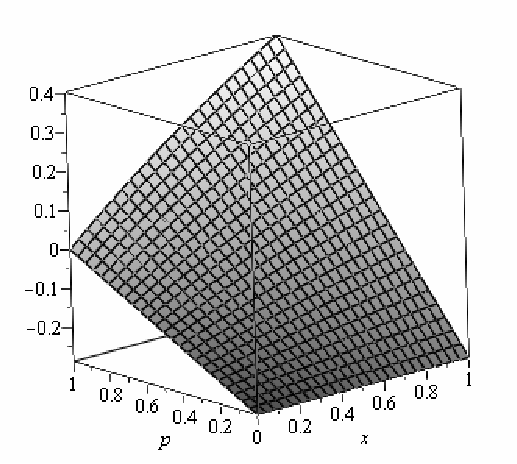
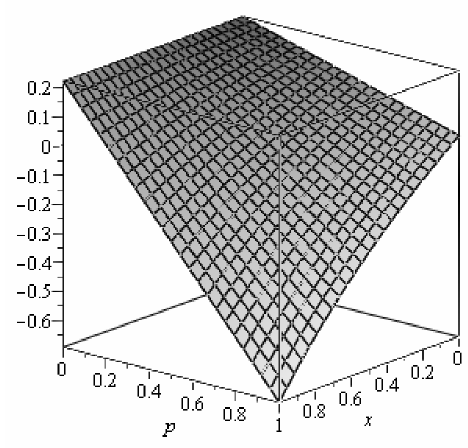
Appendix-1 Kelly Betting and Asymmetric Payoffs

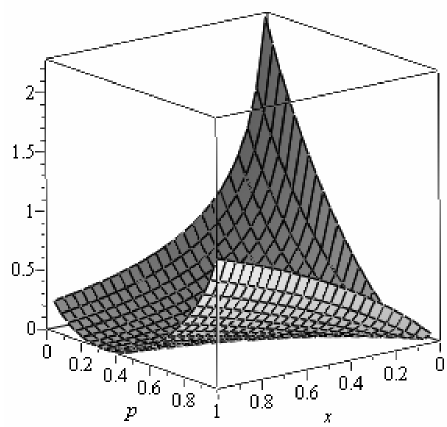
Bet Size=50% and y=0.5



Bet Size=-50% and y=0.5



Bet Size=Optimal and y=0.5

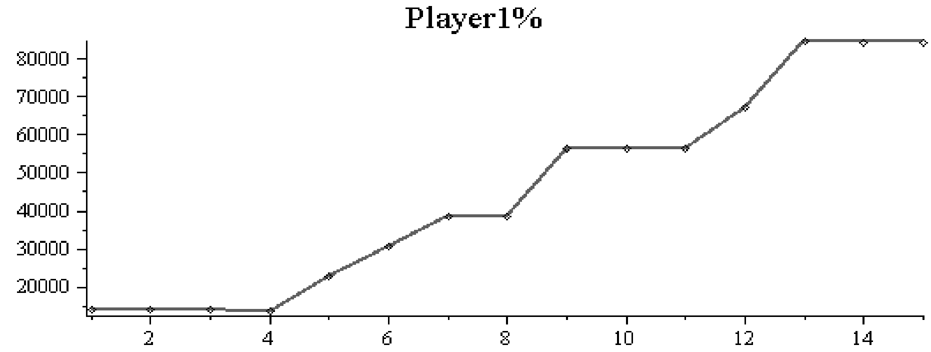


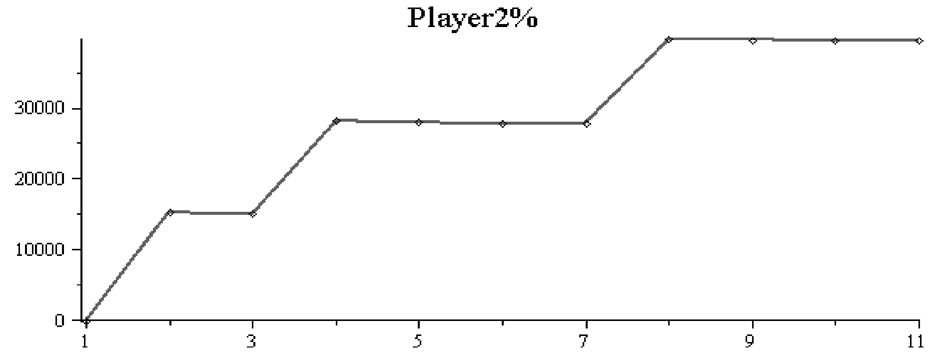
Appendix-2 Fixed vs. Variable Position Sizing

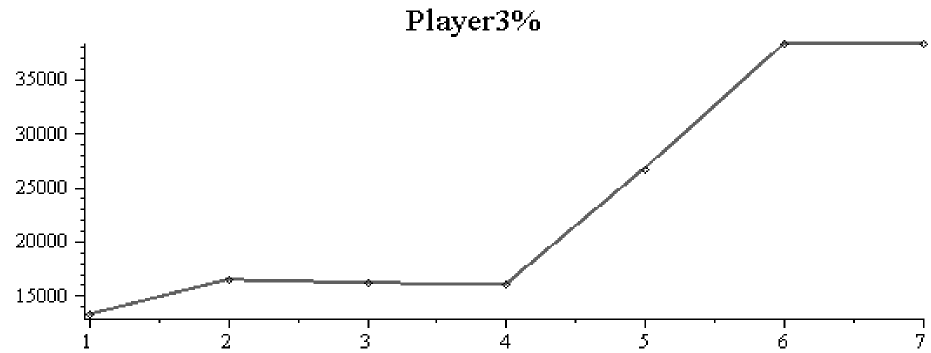
Over time the best way to calculate the position size is based upon the initial portfolio value instead of the portfolio value at each point in time. The reason for that is because the position size remains the same over time i.e. it is not easy buying 2.1 stocks etc. The problem is the percentage loss is increasing over time but only marginally

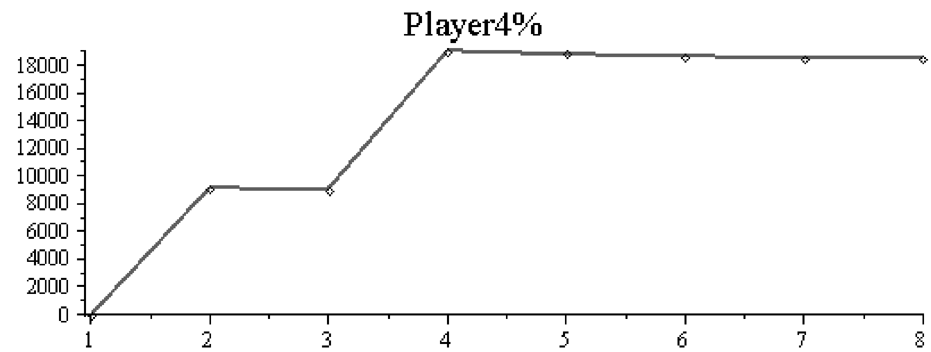
Risk per Share	5			
Number of Sequential Losses	**Portfolio Value**	**Buy Stock**	**Loss**	**Percentage Loss**
1	10000.00	20.00	100.00	0.0100
2	9900.00	20.00	100.00	0.0101
…	…	…	…	…
5	9600.00	20.00	100.00	0.0104
6	9500.00	20.00	100.00	0.0105
7	9400.00	20.00	100.00	0.0106
…	…	…	…	…
17	8400.00	20.00	100.00	0.0119
18	8300.00	20.00	100.00	0.0120
19	8200.00	20.00	100.00	0.0122
20	8100.00	20.00	100.00	0.0123
…	…	…	…	…
40	6100.00	20.00	100.00	0.0163
51	5000.00	20.00	100.00	0.0200
81	4900.00	20.00	100.00	0.0204
98	300.00	20.00	100.00	0.3333
99	200.00	20.00	100.00	0.5000
100	100.00	20.00	100.00	1
Number of Sequential Losses	**Portfolio Value**	**Buy Stock**	**Loss**	**Percentage Loss**
1	10000.00	20.00	100.00	0.01
2	9900.00	19.80	99.00	0.01
…	…	…	…	…
5	9605.96	19.21	96.06	0.01
6	9509.90	19.02	95.10	0.01
7	9414.80	18.83	94.15	0.01
…	…	…	…	…
17	8514.58	17.03	85.15	0.01
18	8429.43	16.86	84.29	0.01
19	8345.14	16.69	83.45	0.01
20	8261.69	16.52	82.62	0.01

Appendix-3 Player1% vs. the Rest









Appendix-4 Probability Distributions and SP-500 Stocks