

# Liquidation in the Face of Adversity: Stealth Vs. Sunshine Trading, Predatory Trading Vs. Liquidity Provision

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## Abstract

We consider a multi-player situation in an illiquid market in which one player tries to liquidate a large portfolio in a short time span, while some competitors know of the seller's intention and try to make a profit by trading in this market over a longer time horizon. We show that the liquidity characteristics, the number of competitors in the market and their trading time horizons determine the optimal strategy for the competitors: they either provide liquidity to the seller, or they prey on her by simultaneous selling. Depending on the expected competitor behavior, it might be sensible for the seller to pre-announce a trading intention ("sunshine trading") or to keep it secret ("stealth trading").

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# Introduction

A variety of circumstances such as a margin call or a stop-loss strategy in combination with a large price drop can force a market participant (the “seller”) to liquidate a large asset position urgently. Such a swift liquidation may result in a significant impact on the asset price. Hence, intuitively it seems to be crucial to prevent information leakage while executing the trade, for informed market participants (the “predators” or the “competitors”) could otherwise try to earn a profit by predatory trading: They can sell in parallel with the seller and cover their short positions later at a lower price. Probably the most widely known example of such a situation is the alleged predation on the hedge fund LTCM<sup>1</sup>. Surprisingly, however, some sellers do not follow a secretive “stealth trading” strategy but rather practice “sunshine trading”, which consists in pre-announcing the trade to competitors so as to attract liquidity<sup>2</sup>.

Our goal in this paper is to propose a new model of a competitive trading environment that explains the tradeoff that leads the seller to choose between stealth and sunshine execution and the competitors to choose between predation and liquidity provision. We argue that these choices are driven by the relations between the different liquidity parameters of the market, the number of competitors of the seller and the trading time horizons. In particular, different behavioral patterns may coexist within the same set of agents when they are trading in markets of different liquidity types. Since our model market is semi-strong efficient and allows for anonymous trading possibilities, our results are applicable to a wide variety of real-world markets including most equity exchange markets.

To fully acknowledge the roles of the different liquidity parameters of the market and of the number of competitors of the seller, we need to relax all exogenous trading constraints in our model. In particular, we do not require that predators face the same time constraint as the seller. This assumption is reasonable as sellers typically must achieve a trading target in a fixed and relatively short time horizon—e.g., a margin call has to be covered by the end of the day—while predators often may afford to maintain a long or a short position for a number of days. In order to capture the structure of this situation, we consider a two stage model of an illiquid market. In the first stage, the seller as well as the predators trade; in the second stage, only the predators trade and unwind the asset positions they acquired during the first stage. Liquidity effects are incorporated into our market model by applying a permanent as well as a temporary impact as in the market model proposed by Almgren and Chriss (2001) and used by Carlin, Lobo, and Viswanathan (2007). For the sake of simplicity, throughout this paper we focus on the liquidation of a long position of assets; equivalent statements hold for the liquidation of a short position.

In our analysis of the optimal agent behavior in this model, we first assume that all agents know the seller’s liquidation intentions. We derive a Nash equilibrium of optimal trading strategies for the seller and the predators, and we show that, in equilibrium, the predators’ optimal strategy depends heavily on the liquidity type of the market. We identify two distinct types of illiquid markets: First, if the temporary price impact dominates the permanent impact then prices show a high resilience after a large transaction. The price in such “elastic” markets behaves similar to a rubber band: trading pressure can stretch it, but after the trading pressure reduces, the price recovers. Such market conditions can occur when it is difficult to find counterparties for a specific deal within a short time. In such a market, the optimal strategy for the predators is to cooperate with the seller: they should buy some of the seller’s assets and sell them at a later point in time. On the other hand, if the permanent price impact of a trade outweighs the temporary impact, then large transactions have a long-lasting influence. In such “plastic” markets, the trading pressure exerts a “plastic deformation” on the market price. Such a situation can arise when a large supply or demand of the asset is interpreted as the revelation of new information on the fundamentals of the asset. Under these conditions the optimal behavior of the predators is the opposite: they should sell in parallel to the seller and buy back at a later point in time (predatory trading). In this case, the price is pushed far down during the first stage and recovers during the second stage, resulting in

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<sup>1</sup>See, e.g., Lowenstein (2001), Jorion (2000) and Cai (2003)).

<sup>2</sup>See, e.g., Harris (1997) and Dia and Pouget (2006). A similar phenomenon occurs in the sometimes widespread distribution of so-called “indications of interest” in which brokers announce tentative conditions for certain liquidity trades.

price overshooting. The latter effect disappears as the number of predators increases; for a large number of predators, the market price incorporates the seller's intentions almost instantly and exhibits little drift thereafter. This effect indicates that our model market fulfills the semi-strong form of the efficient markets hypothesis.

Through sunshine trading, the seller can increase the number of predators. We find that in elastic markets, the seller always achieves a higher return when predators are participating than when she is selling by herself. Therefore, sunshine trading appears to be sensible in such a market. In a plastic market, the seller's return can be significantly reduced by predators; however, as the number of predators increases, the optimal strategy for the predators changes from predation to cooperation and the return for the seller increases back, sometimes even above the level of return obtained in the absence of predators. Hence, if the seller has reason to believe that there is *some* leakage of information<sup>3</sup>, it may be sensible to take the initiative of publicly announcing the impending trade so as to turn around the adverse situation of predation by few competitors into the beneficial situation of liquidity provision.

Although our approach is normative rather than descriptive, our model provides a number of empirically testable hypotheses for both seller and competitor behavior. In our model, sunshine trading is rational in elastic markets or when the trading horizon of the seller is comparatively short. We therefore suspect that sunshine trades and indications of interest are usually short-term and occur in markets with high temporary impact, while we conjecture that efforts to conceal trading intentions are particularly strong in plastic markets.

We predict that competitors in plastic markets pursue predatory trading if they know about selling intentions of other agents, while we expect them to provide liquidity in elastic markets. Unfortunately, we are not aware of any systematic study of informed competitors reactions to trading under varying market liquidity<sup>4</sup>. However, the analysis of distressed hedge funds lends anecdotal support to our hypothesis. During the LTCM crisis in 1998, several competitors allegedly engaged in front-running and predatory trading, while no individual investor was willing to acquire LTCM's positions and thus provide liquidity. According to our results, such a behavior is rational in plastic markets. The price evolution after the LTCM crisis indicates that its liquidation had a predominantly permanent effect<sup>5</sup>, i.e., that the market was indeed plastic.

More recently, the hedge fund Amaranth experienced severe losses resulting in the need for urgent liquidation<sup>6</sup>. Contrary to LTCM, Amaranth quickly found a buyer for its portfolio<sup>7</sup>. In the Amaranth case, liquidity provision apparently appeared as the more profitable option for competitors compared to predatory trading. How can the differences between competitors' behavior in the LTCM and Amaranth cases be explained? In both cases very large market participants were in distress, promising large profit opportunities for competitors. However, Amaranth operated in the natural gas market, which behaved

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<sup>3</sup>In practice, information leakage can occur due to a variety of circumstances. For instance, as in the case of the LTCM crisis, the position may simply be too large to keep its liquidation secret. In a much more common situation, the execution of the trade will be commissioned to an investment bank, but advance price quotes are obtained from several banks. Banks that are not successful in bidding for the trade will nevertheless be informed about its existence and hence constitute potential predators. When obtaining price quotes, it is therefore common practice for the client to distribute only a limited amount of information on their "bid sheets" so as to reduce the potentially adverse effects of predatory trading. Another example is provided by market makers who must report large transactions.

<sup>4</sup>This could be carried out, e.g., by analyzing the order flow after pre-announcement of a sale. In plastic markets, we expect to see an initial increase of additional seller initiated trades. In elastic markets, we expect to see an increase in buy orders.

<sup>5</sup>Lowenstein (2001) notes that (Epilogue, page 229): "(...) a year after the bailout [of LTCM], swap spreads remained (...) far higher than when Long-Term had entered the (...) trade."

<sup>6</sup>For a description of the Amaranth case, see Till (2006) and Chincarini (2007). Finger (2006) finds that "The events of September [2006] led to the greatest losses ever by a single hedge fund, close to twice the money lost by Long Term Capital Management."

<sup>7</sup>Till (2006) notes that "Amaranth sold its entire energy-trading portfolio to J.P. Morgan Chase and Citidel Investment Group on Wednesday, September 20th [2006]."

elastic during a previous hedge fund liquidation<sup>8</sup>. According to our model liquidity provision is the most profitable behavior in such an elastic market.

The profitability of liquidity provision in elastic markets is confirmed by Coval and Stafford (2006), who find that providing liquidity to open-ended mutual funds that suffer severe cash outflows promises average annual abnormal excess returns well over 10%. This supports our hypothesis since these profits are made on the temporary nature of the price impact. Interestingly, the impact of stock sales in markets that do not suffer from extreme cash outflows appears to be predominantly permanent, resulting in profitable predatory trading opportunities for insiders.

Our research builds on previous work in three research areas. The first area to which our work is connected is research on predatory trading. In previous studies, the size of the liquidation completely determines the optimal action of the competitors. In these models, predatory trading is always optimal for large liquidations. For small liquidations, predatory trading is always or never optimal, depending of the model at hand.

Brunnermeier and Pedersen (2005) suggest a model in which the total rate of trading as well as the asset positions of all traders face exogenous constraints. They show that in equilibrium in their model predation and price overshooting occur necessarily, irrespective of the market environment<sup>9</sup>. As a side effect of the exogenous trading constraint, their model market is weakly inefficient: even if the number of informed predators is large, the market price changes continuously in reaction to the trading of the seller and the predators.

Carlin, Lobo, and Viswanathan (2007) propose a model in which competitors can engage in and refrain from predatory trading, however there is no room for optimal liquidity provision. To explain abstinence from predatory trading, they assume that all market participants repeatedly execute large transactions in a fully transparent market<sup>10</sup>; in such a repeated game, predation can be punished by applying a tit-for-tat strategy. In their model, competitors always refrain from predatory trading while others are liquidating small positions, but cooperation always breaks down if an unusually large distressed sale is occurring. Although their analysis of a one stage game is also at the foundation of our model, the two models diverge in their qualitative predictions of trading decisions: their model predicts that predatory trading is most widespread in elastic markets, while our model predicts the opposite.

Attari, Mello, and Ruckes (2005) discuss trading strategies against a financially constraint arbitrageur. Price impact in their model market is completely temporary, resulting in an elastic market with profitable liquidity provision. By clever exploitation of the arbitrageur's capital constraint, the competitors can profitably engage in predatory trading, but only for arbitrageurs with very large asset positions.

In a second line of research, the effects of sunshine trading are investigated. In a theoretical investigation, Admati and Pfleiderer (1991) propose a model in which sunshine trading is always increasing the seller's return as long as speculators do not face market entry costs. The underlying motives for sunshine trading in this model and in our model are very different<sup>11</sup>. Empirical evidence on the benefit of trade pre-announcements appears to be mixed (see, e.g., Harris (1997), Dia and Pouget (2006)), which

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<sup>8</sup>Till (2006) observes that "There was a preview of the intense liquidation pressure on the Natural Gas curve on 8/2/06, the day before the [natural-gas-oriented] energy hedge fund, MotherRock, announced that they were shutting down. (...) A near-month calendar spread in Natural Gas experienced a 4.5 standard-deviation move intraday before the spread market normalized by the close of trading on 8/2/06."

<sup>9</sup>The only situation in which predatory trading does not occur in the model of Brunnermeier and Pedersen (2005) is when there is significant capacity on the sideline. In their model, this implies that the asset is heavily undervalued. They show that this cannot be the case in equilibrium.

<sup>10</sup>Our model explains cooperation in a different way; in particular, our model is also applicable to anonymous markets.

<sup>11</sup>In the model of Admati and Pfleiderer (1991), sunshine traders can expect to obtain better trade conditions in the market since it is assumed that their actions are not based on private information. In our model, we do not assume that sunshine trades have a special motivation; instead, we show that sunshine trading under certain market conditions can raise the attention of competitors and attract them to provide liquidity. A different market perception of sunshine trades can easily be incorporated in our framework by applying different liquidity parameters for sunshine trades and for unannounced trades.

is in line with our observation that the potential benefit of sunshine trading depends on the liquidity characteristics of the market.

The third line of research consists of empirical investigations and theoretical modeling of the market impact of large transactions. The empirical literature is extensive<sup>12</sup>. These empirical results, most notably the identification of temporary and permanent impact, have led to theoretical models of illiquid markets. One line of research focused on deriving the underlying mechanisms for these liquidity effects<sup>13</sup>. A second line takes the liquidity effects as exogenously given and derives optimal trading strategies within such an idealized model market. We follow this second approach and apply a market model similar to the one proposed by Almgren and Chriss (2001). Several alternative models have been proposed<sup>14</sup>; the advantages and disadvantages of these models are still a topic of ongoing research.

The remainder of this paper is structured as follows. In Section I, we introduce the market model and specify the game theoretic optimization problem. As a preparation for the general two stage model, we review predation in a one stage model in Section II. In this model, the seller and the predators face the same time constraint, i.e., the predators do not have the opportunity to trade after the seller finished selling. In the main Section III, we turn to the more general two stage framework and derive our main results. After identifying the Nash equilibrium of optimal trading strategies in Section IV, we investigate the qualitative properties of our model in three example markets in Section V. Thereafter, we summarize the general properties in Section VI. Section VII concludes. Appendix A contains additional propositions on the one stage model. All proofs of propositions are given in Appendix B.

## I The market model

We start by describing the market dynamics and trade motives of market participants. The market consists of a risk-free asset and a risky asset. Trading takes place in continuous time. We assume that the risk-free asset does not generate interest. In this market we consider  $n + 1$  strategic players and a number of noise traders. The strategic players are aware of liquidity needs in the market and optimize their trading to profit from these needs, whereas noise traders have less information and trade based on exogenous liquidity and investment needs. We assume that the number of strategic players ( $n + 1$ ) is given a priori. During our analysis, we will perform comparative statics and discuss the incentives for each player to change the number of strategic players in the market.

We denote the time-dependent risky asset positions of the strategic players by  $X_0(t), X_1(t), \dots, X_n(t)$  and assume that they are differentiable in  $t$ . Their trading  $\dot{X}_i(t)$  affects the market price in the form of a permanent impact and a temporary impact. Trades at time  $t$  are thus executed at the price

$$P(t) = \tilde{P}(t) + \gamma \sum_{i=0}^n (X_i(t) - X_i(0)) + \lambda \sum_{i=0}^n \dot{X}_i(t). \quad (1)$$

Here,  $\tilde{P}(t)$  is a one-dimensional arithmetic Brownian motion without drift, starting at  $\tilde{P}(0) = P_0$  and defined on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . This term reflects the price changes due to the random trades of noise traders. The second term on the right hand side represents the permanent price impact resulting

<sup>12</sup>See, e.g., Kraus and Stoll (1972), Holthausen, Leftwich, and Mayers (1987), Holthausen, Leftwich, and Mayers (1990), Barclay and Warner (1993), Chan and Lakonishok (1995), Biais, Hillion, and Spatt (1995), Kempf and Korn (1999), Chordia, Roll, and Subrahmanyam (2001), Chakravarty (2001), Lillo, Farmer, and Mantegna (2003), Mönch (2004), Almgren, Thum, Hauptmann, and Li (2005), Coval and Stafford (2006), Obizhaeva (2007), Large (2007).

<sup>13</sup>See for example Kyle (1985), Glosten and Milgrom (1985), Easley and O'Hara (1987), Foster and Viswanathan (1996), Frey (1997), O'Hara (1998), Bondarenko (2001) and Biais, Glosten, and Spatt (2005).

<sup>14</sup>See, e.g., Bertsimas and Lo (1998), Almgren (2003), Butenko, Golodnikov, and Uryasev (2005), Obizhaeva and Wang (2006), Engle and Ferstenberg (2006), Alfonsi, Schied, and Schulz (2007), Frey (1997), Frey and Patie (2002), Bank and Baum (2004), Çetin, Jarrow, and Protter (2004), Çetin, Jarrow, Protter, and Warachka (2006), Jarrow and Protter (2007).

from the change in total asset position of all strategic players. Its magnitude is determined by the parameter  $\gamma > 0$ . The third term reflects the temporary impact caused by the net trading speed of all strategic investors. Its magnitude is controlled by the parameter  $\lambda > 0$ . This price dynamics model is a multi-player extension of the framework introduced by Almgren and Chriss (1999) and Almgren (2003) with linear permanent and linear temporary impact.

In this market, the strategic players are facing the following optimization problem. Each player  $i$  knows all other players' initial asset positions  $X_j(0)$  and their target asset positions  $X_j(T)$  for some fixed point  $T > 0$  in the future<sup>15</sup>. We assume that these trading targets are binding; players are not allowed to violate their targets. We assume that all players are risk-neutral<sup>16</sup>; therefore, players want to maximize their own *expected* return by choosing an optimal trading strategy  $X_i(t)$  given their boundary constraints on  $X_i(0)$  and  $X_i(T)$ . In mathematical terms, each player is looking for a strategy that realizes the maximum

$$r_i := \max_{X_i} \mathbb{E}(\text{Return for player } i) = \max_{X_i} \mathbb{E} \left( \int_0^T (-\dot{X}_i(t)) P(t) dt \right) \quad (2)$$

$$= \max_{X_i} \mathbb{E} \left( - \int_0^T \dot{X}_i(t) \left( \tilde{P}(t) + \gamma \sum_{j=0}^n (X_j(t) - X_j(0)) + \lambda \sum_{j=0}^n \dot{X}_j(t) \right) dt \right). \quad (3)$$

Although in principle the strategies  $X_i$  might be predictable, we limit our discussion to deterministic strategies, where the function  $X_i$  does not depend on the stochastic price component  $\tilde{P}(t)$ . In such open-loop strategies, all players determine their trade schedules *ex ante*<sup>17</sup>. Hence,

$$r_i = \max_{X_i} \left( - \int_0^T \dot{X}_i(t) \left( P_0 + \gamma \sum_{j=0}^n (X_j(t) - X_j(0)) + \lambda \sum_{j=0}^n \dot{X}_j(t) \right) dt \right). \quad (4)$$

A set of strategies  $(X_0, X_1, \dots, X_n)$  satisfying Equation (4) for all  $i = 0, 1, \dots, n$  constitutes a Nash equilibrium; we call such a set of strategies *optimal*<sup>18</sup> and denote the corresponding optimal returns in equilibrium by  $R_i := r_i$ . These are determined by the *expected* price

$$\bar{P}(t) := \mathbb{E}(P(t)) = P_0 + \gamma \sum_{i=0}^n (X_i(t) - X_i(0)) + \lambda \sum_{i=0}^n \dot{X}_i(t). \quad (5)$$

Whenever we refer to *price* or *return* in the following, we will always refer to the *expected price*  $\bar{P}(t)$  and the *expected return*  $-\int \dot{X}_i(t) \bar{P}(t) dt$ .

## II The one stage model

In this section, we investigate the optimal strategies in a one stage framework: *all* players trade over the same time interval  $[0, T_1]$ . The results in this section will be used in the analysis of a two stage model in the following sections.

<sup>15</sup>For the purposes of this paper, we assume that all strategic players have perfect information. For imperfect information, we expect to obtain slightly changed dynamics (potentially including a “waiting game” as in Foster and Viswanathan (1996)), but expect the qualitative results on predatory trading and liquidity provision to remain unchanged.

<sup>16</sup>See also Footnote 22.

<sup>17</sup>The analysis of closed-loop strategies in which players can dynamically react to other players actions is mathematically more difficult. It is often not possible to derive closed form solutions, on which we rely in the proof of Theorem 2. Carlin, Lobo, and Viswanathan (2007) show numerically that closed-loop solutions of the one stage model (see Section II) are similar to the open-loop solutions and do not exhibit any new qualitative features. Therefore, no major differences are expected in the two stage model introduced in Section III.

<sup>18</sup>These strategies remain optimal for the entire trading time. At a future point in time  $t \in [0, T]$ , there is no reason to deviate from the trade schedule chosen at time 0 as long as no other player deviated from her trade schedule.

Parameter	Value
Asset position $X_0$	1
Initial price $P_0$	10
Duration $T_1$	1
Permanent impact sensitivity $\gamma$	3
Temporary impact sensitivity $\lambda$	1

Table I: Parameter values used for numerical computation of the figures in Section II.

The optimal strategies in the one stage framework were derived by Carlin, Lobo, and Viswanathan (2007). We repeat their result:

**Theorem 1** (Carlin, Lobo, and Viswanathan (2007)). *Assume that  $n + 1$  players are trading simultaneously in a time period  $t \in [0, T_1]$ . They start with asset positions  $X_i(0)$  and need to achieve a target asset position  $X_i(T_1)$ . Furthermore, these players are risk-neutral and are aware of all other players' asset positions and trading targets. Then the unique optimal strategies for these  $n + 1$  players (in the sense of a Nash equilibrium) are given by:*

$$\dot{X}_i(t) = ae^{-\frac{n}{n+2}\frac{\gamma}{\lambda}t} + b_i e^{\frac{\gamma}{\lambda}t} \quad (6)$$

with

$$a = \frac{n}{n+2} \frac{\gamma}{\lambda} \left(1 - e^{-\frac{n}{n+2}\frac{\gamma}{\lambda}T_1}\right)^{-1} \frac{\sum_{i=0}^n (X_i(T_1) - X_i(0))}{n+1} \quad (7)$$

$$b_i = \frac{\gamma}{\lambda} \left(e^{\frac{\gamma}{\lambda}T_1} - 1\right)^{-1} \left(X_i(T_1) - X_i(0) - \frac{\sum_{j=0}^n (X_j(T_1) - X_j(0))}{n+1}\right). \quad (8)$$

*Proof.* See Carlin, Lobo, and Viswanathan (2007). □

For the rest of this section, we consider the following more specific situation: One player (say player 0) wants to sell an asset position  $X_0(0) = X_0$  in the time interval  $[0, T_1]$ , i.e. the target is given by  $X_0(T_1) = 0$ . All other players (i.e., players 1, 2, ...,  $n$ ) do not want to change their initial and terminal asset positions (for simplicity, we assume that  $X_i(0) = X_i(T_1) = 0$  for  $i \neq 0$ ), but they want to exploit their knowledge of player 0's sales.

The result is *preying* of the  $n$  players on the first player (see Figure 1 and 2; see Table I for the parameter values used for the figures): while the first player is starting to sell off her asset position, the other players sell short the asset and realize a comparatively high price per share. At the end of the trading period, the price has been pushed down by the combined sales of both seller and predators. While the seller liquidates the remaining part of her long position at a fairly low price, the other players can now close their short positions at a favorable price. Since the price has dropped, the preying players need to spend less on average for buying back than they received for initially selling short. In the following, we refer to player 0 as the “seller” and to the players 1, 2, ...,  $n$  as the “predators”.

In the one stage model considered so far, there is no room for cooperation; preying **always** occurs. The seller's return is further deteriorating as the number of predators increases; preying becomes more competitive with more players being involved (see Figure 3). We will see in the next section that relaxing the exogenous time constraint on the positions of predators can lead to a more differentiated behavior. It includes in particular the possibility of liquidity provision to the seller.

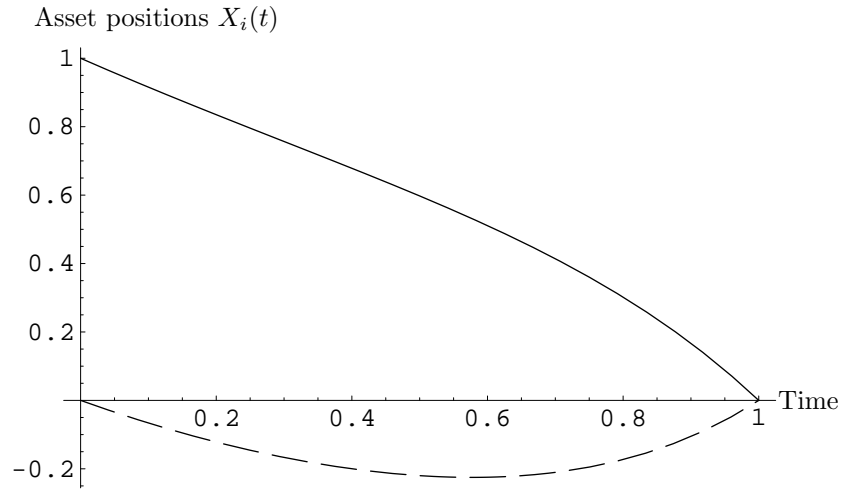


Figure 1: Asset positions  $X_i(t)$  over time. The solid line represents the seller, the dashed line the predator ( $n = 1$ ).

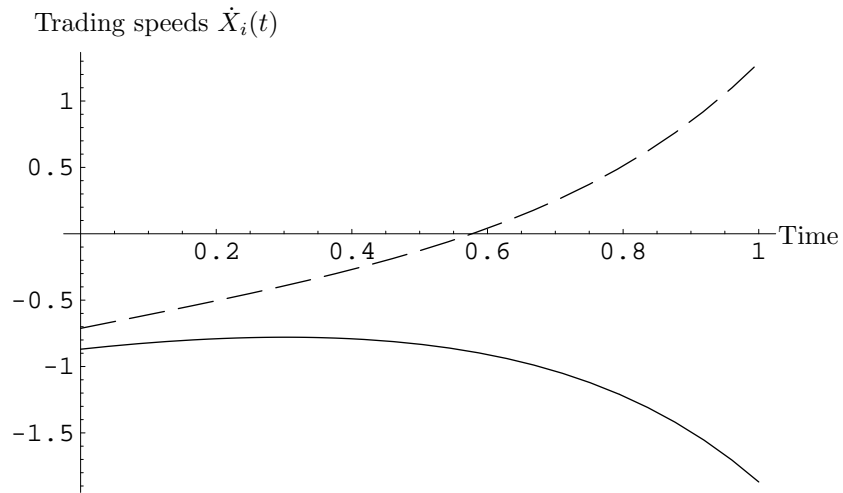


Figure 2: Trading speeds  $\dot{X}_i(t)$  over time. The solid line represents the seller, the dashed line the predator ( $n = 1$ ).



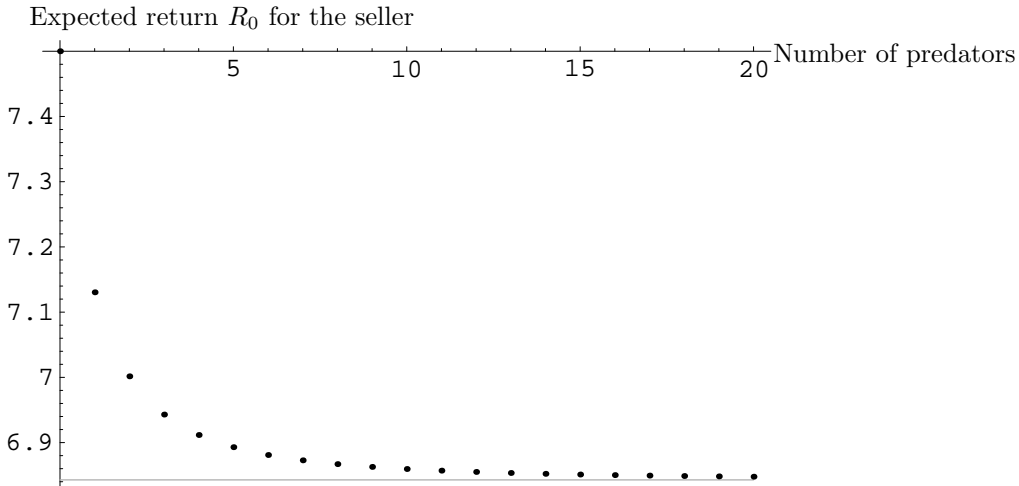


Figure 3: Expected cash return for the seller (player 0) from selling  $X_0$  shares, depending on the number  $n$  of predators. The expected return in absence of predators is 7.5 (intersection point of x- and y-axes). The grey line at the bottom corresponds to the limit  $n \rightarrow \infty$ .

### III The two stage model

In the previous section, we have assumed that the seller and the predators are limited to trade during the same time interval. As we have mentioned earlier, in reality the seller is often facing a stricter time constraint than the predators do. While the seller usually needs to liquidate her asset position within a few hours, the predators can often afford to close their positions at a later point in time. In the following, we therefore extend the one stage model considered so far to a two stage framework<sup>19</sup> and assume that:

- In stage 1, *all* players (the seller and the predators) are trading.
- In stage 2, *only* the predators are trading; the seller is not active.

The first stage runs from  $t = 0$  to  $T_1$ , the second stage<sup>20</sup> from  $T_1$  to  $T_2$ . The asset position of player  $i$  is denoted by  $X_i(t)$  with  $t \in [0, T_2]$ . We require the strategies  $X_i(t)$  to be differentiable within each stage, but they need not be differentiable at  $t = T_1$ .

The market prices are governed by

$$P(t) = \tilde{P}(t) + \gamma \sum_{i=0}^n (X_i(t) - X_i(0)) + \lambda \sum_{i=0}^n \dot{X}_i(t) \quad (9)$$

<sup>19</sup>The framework can be extended further to a three stage model including a stage 0 in which only the predators are allowed to trade. Such a setup can capture the effects of front-running, which results in different results in particular for price overshooting. We limit our analysis to the two stage model since in most practical cases, there is little room for front-running due to legal constraints or insufficient time (i.e., stage 0 is very short); see the introduction for examples. As another alternative, the model can account for a different trading horizon for each predator. This increases the mathematical complexity, but does not lead to qualitatively new phenomena within stage 1.

<sup>20</sup>In reality, the seller usually has to liquidate an asset position by the end of the trading day. In this case, the second stage begins at the open of the next trading day. Our framework can easily be extended to accommodate for this setting by having the second stage run from  $\hat{T}_1 > T_1$  to  $T_2$ . Since we assumed that the seller and the predators are risk-neutral, this does not change any of the statements in this exposition; for notational simplicity, we therefore restrict ourselves to the case where the second stage starts immediately after the first stage.

for  $t \in [0, T_2] \setminus T_1$ . Again,  $\tilde{P}(t)$  is an arithmetic Brownian motion without drift, starting at  $\tilde{P}(0) = P_0$ . Since the  $X_i(t)$  might be non-differentiable at  $t = T_1$ , the above formula might not be well-defined; we therefore set

$$P(T_1) = \lim_{t \searrow T_1} P(t), \quad P(T_1-) := \lim_{t \nearrow T_1} P(t), \quad (10)$$

forcing the price to be right-continuous.

The seller (player 0) is assumed to liquidate an asset position  $X_0 = X_0(0)$  during stage 1:  $X_0(t) = 0$  for all  $t \in [T_1, T_2]$ . We assume that the  $n$  predators want to exploit their knowledge of the seller's intentions, but do not want to change their asset position permanently. We therefore require that the predators have the same asset positions at the beginning of stage 1 and at the end of stage 2:  $X_i(0) = X_i(T_2)$ . For notational simplicity, we assume<sup>21</sup>  $X_i(0) = 0$ . All assumptions and notation introduced in Section I apply in our two stage model; in particular, we restrict our analysis to risk-neutral players<sup>22</sup> following deterministic strategies.

There are no a-priori restrictions on predators' asset positions  $X_i(T_1)$  at the end of stage 1. They can be positive, i.e., the predators buy some of the seller's shares in stage 1 and thereby provide liquidity to the seller. Alternatively, they can be negative, i.e., the predators sell parallel to the seller, driving the market price further down and preying on the seller. In the next section, we show that the occurrence of liquidity provision or predation depends on the market characteristics, in particular on the balance between temporary and permanent impact.

## IV Optimal strategies in the two stage model

We can now describe the optimal behavior of all  $n + 1$  strategic players in the two stage model introduced in the previous section. If the optimal asset positions  $X_i(T_1)$  of the predators at the end of stage 1 are known, the entire optimal strategies are determined by Theorem 1: In stage 1,  $n + 1$  players are trading and the initial and final asset positions are known; in stage 2,  $n$  players are trading and again the initial and final asset positions are known<sup>23</sup>. Therefore, we only need to derive the optimal asset positions<sup>24</sup>  $X_i(T_1)$  for all predators  $i = 1, 2, \dots, n$  (see Figure 4 for an illustration).

**Theorem 2.** *In the unique Nash equilibrium, all predators acquire the same asset position during stage 1:*

$$X_i(T_1) = F\left(\frac{\gamma T_1}{\lambda}, \frac{T_2}{T_1}, n\right) X_0. \quad (11)$$

The function  $F$  is given in closed form in the proof in Appendix B. For the special case  $n = 1$ , we obtain

$$X_1(T_1) = -\frac{\left(-2 - e^{\frac{\gamma T_1}{3\lambda}} - e^{\frac{2\gamma T_1}{3\lambda}} + e^{\frac{\gamma T_1}{\lambda}}\right) \frac{\gamma}{\lambda} (T_2 - T_1)}{6\left(-1 + e^{\frac{\gamma T_1}{\lambda}}\right) + \left(2 + e^{\frac{\gamma T_1}{3\lambda}} + e^{\frac{2\gamma T_1}{3\lambda}} + 2e^{\frac{\gamma T_1}{\lambda}}\right) \frac{\gamma}{\lambda} (T_2 - T_1)} X_0. \quad (12)$$

<sup>21</sup>The optimal trading speed  $\dot{X}_i(t)$  of the predators is independent of their initial asset position  $X_i(0)$ . In particular, our results also hold in the case where predators have different initial asset positions.

<sup>22</sup>Risk aversion can be incorporated in two different ways. The first is to regard the different execution time frame of the seller and the predators as proxies of their risk aversion. This provides a simple model of a highly risk averse seller in a market environment with relatively risk-neutral competitors. Alternatively, risk aversion can explicitly be modeled by introducing utility functions for the seller and the predators. This leads to the coexistence of liquidity provision and preying already in the one stage model introduced in Section II. The dynamics for a risk averse seller facing relatively risk-neutral predators is qualitatively very similar to the two stage model presented here. A detailed discussion of the effects of risk aversion lie beyond the scope of this paper and are subject of ongoing research.

<sup>23</sup>In the case  $n = 1$ , it follows from the results in Almgren and Chriss (2001) and Almgren (2003) that the optimal trading strategy in stage 2 is a linear increase / decrease of the predator's asset position.

<sup>24</sup>Carlin, Lobo, and Viswanathan (2007) noted this for the single predator case. They also conjectured that in a two stage model there will be price overshooting. As we will see in Section V and Proposition 9, the source of this price overshooting is not necessarily the presence of strategic players. In fact, price overshooting is reduced by predators in elastic markets.

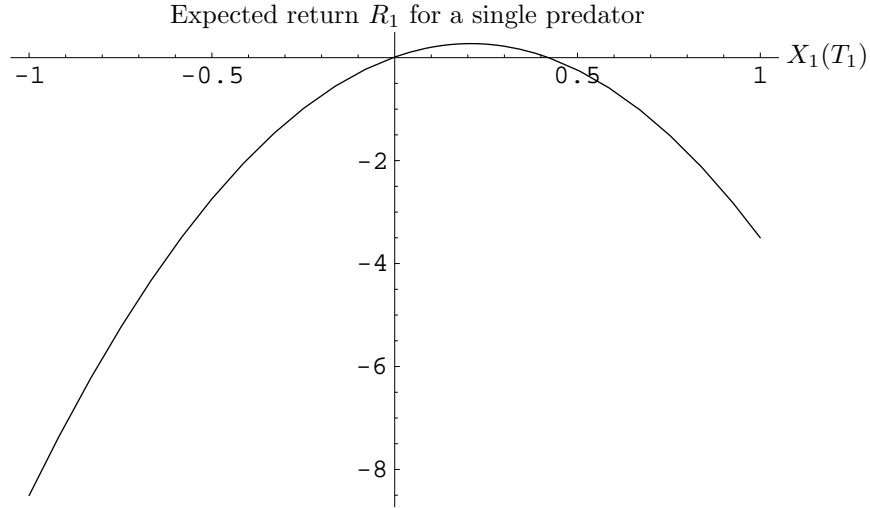


Figure 4: Expected return  $R_1$  for a single predator depending on her asset position  $X_1(T_1)$ . Optimal trading within stage 1 and stage 2 is assumed. Parameters are chosen as in the elastic market in Table II.

Formulas (11) and (12) do not depend on  $\gamma$  and  $\lambda$  separately, but only on the fraction <sup>25</sup>  $\frac{\gamma T_1}{\lambda} = \frac{\gamma}{\lambda/T_1}$ , which can be interpreted as a normalized ratio of liquidity parameters. The permanent impact parameter  $\gamma$  has unit “dollars per share” and is independent of the time unit. The temporary impact parameter  $\lambda$  has unit “dollars per share per time unit” and thus depends on the time unit. The fraction  $\lambda/T_1$  can be interpreted as the temporary impact parameter normalized to the length of the first stage.

In the next section, we will analyze the qualitative influence of the ratio  $\frac{\gamma T_1}{\lambda}$  by reviewing some specific example markets. For notational simplicity, we will implicitly assume that  $T_1 = 1$  and thus restrict our discussion to  $\gamma$  and  $\lambda$ . We will return to the general situation again in Section VI.

## V Example markets

### A Definition of the example markets

In an illiquid market, each market order causes a price impact. Some part of this initial price impact is temporary and therefore vanishes after the execution of the market order. In the following, we will analyze two polar market extremes in more detail:

- **Elastic markets**, in which the major part of the initial total market impact vanishes after the execution of a market order (i.e., temporary impact  $\lambda \gg$  permanent impact  $\gamma$ ). The market price in such markets behaves similar to an elastic rubber band: trading pressure can stretch it, but after the trading pressure reduces, the price recovers.
- **Plastic markets**, in which the price impact of market orders is predominantly permanent (i.e., permanent impact  $\gamma \gg$  temporary impact  $\lambda$ ). In such markets, the trading pressure exerts a “plastic deformation” on the market price.

<sup>25</sup>Since the dependence of  $F$  on  $n$  is non-reciprocal, the joint strategy of the predators changes as the number of predators increases (see also the dependence on  $n$  in Theorem 1), resulting in a reduced joint profit of the predators. Hence, the predators have an incentive to collude.

Parameter	Elastic market	Plastic market	Intermediate market
Asset position $X_0$	1		
Initial price $P_0$	10		
Duration $T_1$ of stage 1	1		
Duration $T_2 - T_1$ of stage 2	1		
Permanent impact sensitivity $\gamma$	1	3	1.8
Temporary impact sensitivity $\lambda$	3	1	1

Table II: Parameter values used for numerical computation in Section V.

Empirical studies report that markets are indeed sometimes plastic and sometimes elastic<sup>26</sup>. In many practical cases however, the market will fall into neither of these two categories, but instead temporary and permanent impact will be balanced; we therefore conclude our case analysis by reviewing an **intermediate market**, that is, a market where temporary and permanent impact are balanced:  $\lambda \approx \gamma$ . For the numerical computations, we used the parameter values given in Table II.

## B Example market 1: Elastic market

To begin with, let us assume that no predators are active in the market. In such a situation, it is optimal for the seller to sell her asset position linearly (Figure 5). We therefore expect that the market price in stage 1 drops dramatically (Figure 6), since in order to satisfy the seller’s trading needs, liquidity is required fast — which is expensive in an elastic market. In stage 2, no selling pressure from the seller exists any more; hence, the market price will bounce back. Furthermore, since the permanent impact is comparatively small, it will bounce back almost completely.

A predator knowing of the seller’s intentions would expect this price pattern. Her natural reaction would therefore be to buy some of the seller’s shares in stage 1 at the very low price and to sell them in stage 2 at the much higher price. Figure 7 shows that this is indeed what happens when the seller and the predators follow their optimal strategies.

As can be seen in these figures, the total asset position  $\sum_{i=1}^n X_i(T_1)$  acquired by the predators at the end of stage 1 increases as the number of predators increases (see also Figure 8). To gain some intuition for this phenomenon, let us assume that  $n_1$  predators optimally acquire a joint asset position of  $n_1 Y_1$  shares. Imagine one of the predators increases her target asset position by 1. This will decrease the profit per share that she makes, but adds another share to her profitable portfolio. If the original target position  $Y_1$  is optimal, then this increase will leave her total profit roughly unchanged:

$$\text{Profit per share} \times 1 - \text{Decrease in profit per share} \times Y_1 \approx 0. \quad (13)$$

Let us now assume that  $n_2 > n_1$  predators are active and that they jointly acquire  $n_1 Y_1$  shares. Now, increasing the target position  $\frac{n_1 Y_1}{n_2}$  of an individual predator by one share changes the predator’s total profit by

$$\text{Profit per share} \times 1 - \text{Decrease in profit per share} \times \frac{n_1 Y_1}{n_2} > 0. \quad (14)$$

---

<sup>26</sup>Holthausen, Leftwich, and Mayers (1987) find that for their data sample, 75% of the total price impact of large transactions was temporary, while the follow-up study (Holthausen, Leftwich, and Mayers 1990) finds that for a different sample, 85% of the total price impact was permanent. Coval and Stafford (2006) show that in markets where investors withdraw their money from open-ended mutual funds, the total price impact of transactions is predominantly temporary, while in other markets the price impact is predominantly permanent. The anecdotal evidence presented in the introduction indicates that the market for derivatives traded by LTCM was plastic, whereas the energy market was elastic during the Amaranth crisis.

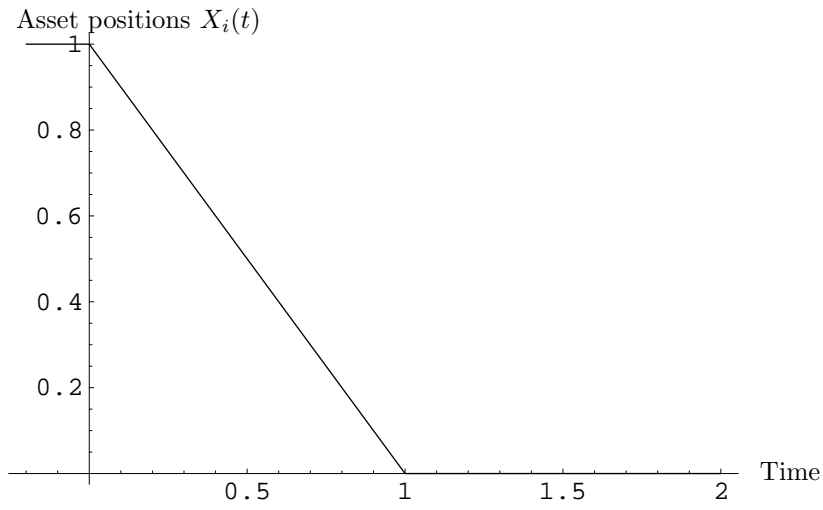


Figure 5: Asset position  $X_0(t)$  of the seller when no predators are active.

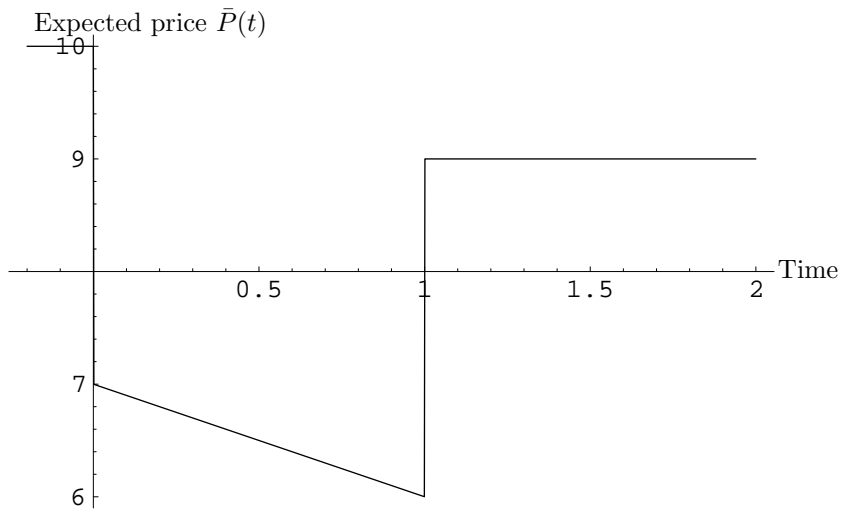


Figure 6: Expected price  $\bar{P}(t)$  in an elastic market over time when no predators are active; at time  $t = 1$ , stage 1 ends and stage 2 begins.

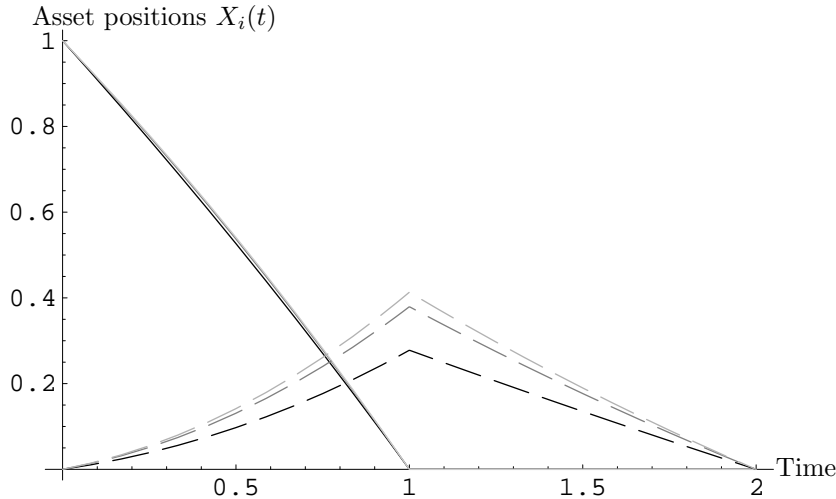


Figure 7: Asset positions  $X_i(t)$  over time in an elastic market; at time  $t = 1$ , stage 1 ends and stage 2 begins. The solid lines represents the seller, the dashed lines the combined asset position of all  $n$  predators. The black lines correspond to  $n = 2$ , the dark grey lines to  $n = 10$  and the light grey lines to  $n = 100$ .

Therefore each predator has an incentive to increase the trading target for the end of stage 1, resulting in an increased joint trading target.

The effect of the predators' trading (buying in stage 1, selling in stage 2) is that prices between stage 1 and stage 2 will even out; the large price jumps expected in the absence of predators will disappear if the number of predators is large enough (see Figure 9). The price overshooting created by the selling pressure of the seller is therefore reduced by the predators.

From the seller's perspective, the predators' trading is beneficial; by buying some of her shares, the predators reduce the seller's market impact and thus increase her return. As we have just discussed, a larger number of predators implies a larger combined purchase by the predators. Hence, the seller can expect to profit from each additional predator, i.e., the larger the number of predators, the larger her profit. This is illustrated by Figure 10; the seller's return is higher when predators are active than it is when there are no predators.

The practical implications are evident: in an elastic market, it is sensible to announce any large, time-constrained asset transaction directly at the beginning of trading in order to attract liquidity.

## C Example market 2: Plastic market

We will now turn to plastic markets, i.e., markets with a permanent impact that considerably exceeds the temporary impact. In such a setting, we expect the price dynamics to be very different from the dynamics described for elastic markets in the previous section.

Let us again assume that no predators are active. Then, the optimal trading strategy for the seller is again a linear decrease of the asset position (see Figure 5). In stage 1, the seller is constantly pushing the market price further and further down; we therefore expect the price to be high at the beginning of stage 1 and low at the end of stage 1 (see Figure 11). In stage 2, the price will bounce back, since the temporary impact of the seller's trading has vanishes. However, this jump will be comparatively small because the temporary price impact is small.

For a predator, this implies that buying some of the seller's shares in stage 1 does not promise any large profit; the price reversion in stage 2 is too small. Instead, it appears more profitable to exploit the

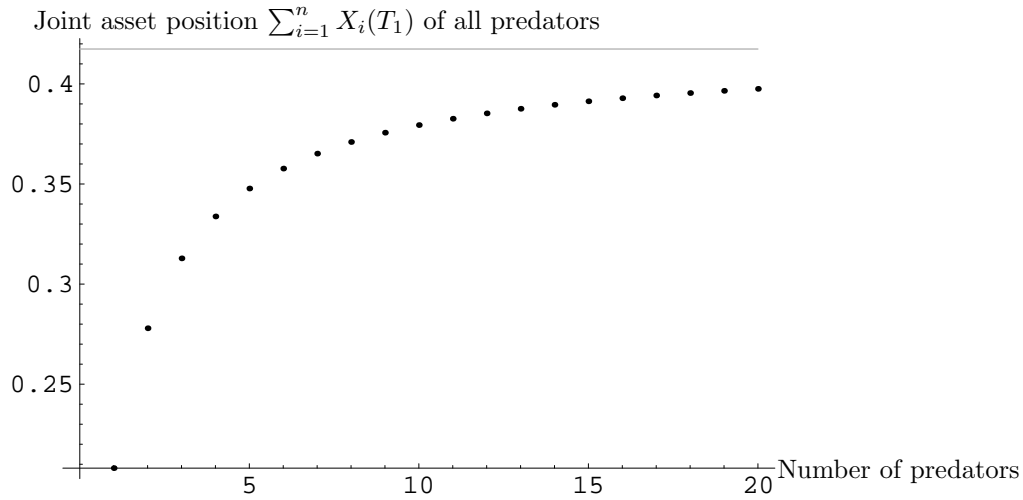


Figure 8: Joint asset position  $\sum_{i=1}^n X_i(T_1)$  of all predators in an elastic market at time  $T_1$  depending on the total number  $n$  of all predators. The grey line represents the limit  $\lim_{n \rightarrow \infty} \sum_{i=1}^n X_i(T_1)$ .

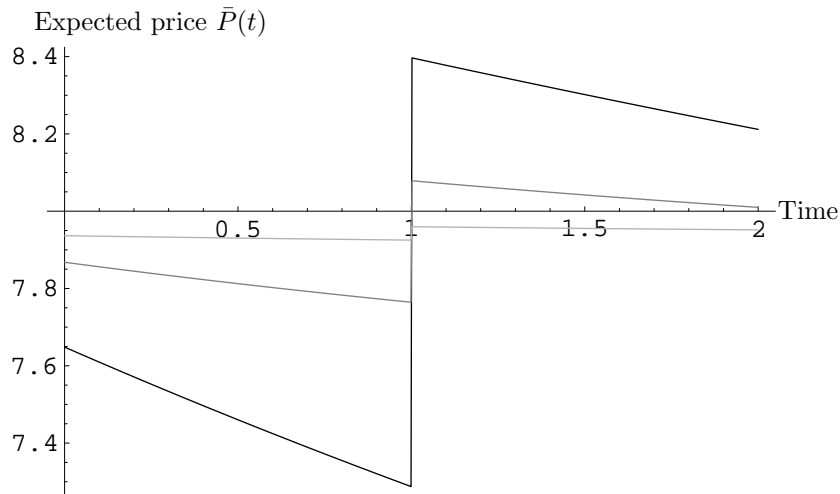


Figure 9: Expected price  $\bar{P}(t)$  in an elastic market over time depending on the number of predators  $n$ ; at time  $t = 1$ , stage 1 ends and stage 2 begins. The black line corresponds to  $n = 2$ , the dark grey line to  $n = 10$  and the light grey line to  $n = 100$ . A significant reduction in price drift can be observed; furthermore,  $\bar{P}(0)$  is smaller than  $P_0 = 10$ .

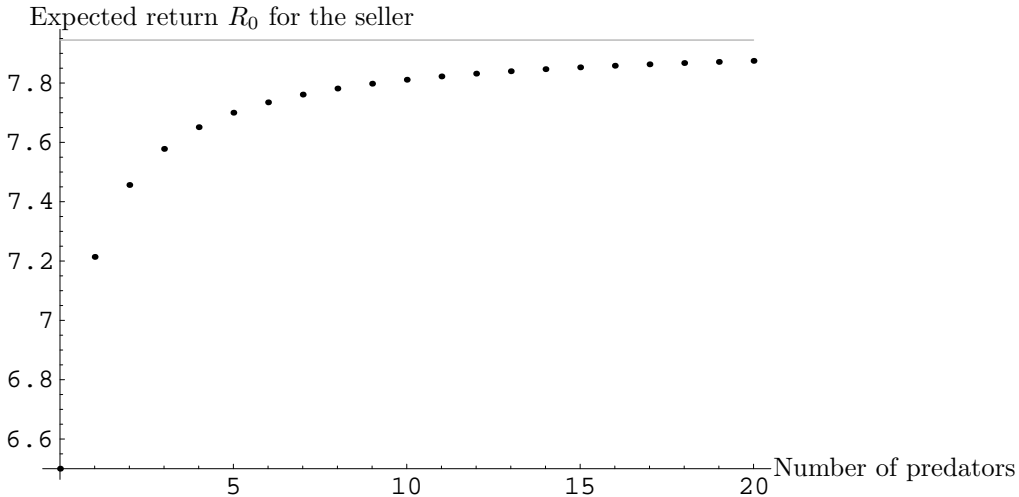


Figure 10: Expected return  $R_0$  for the seller in an elastic market, depending on the number of predators. The grey line represents the limit  $n \rightarrow \infty$ . The return for the seller without predators is at the intersection of  $x$ - and  $y$ -axis.

price changes *within* stage 1 instead of the price changes *between* stage 1 and stage 2. By selling short the asset at the beginning of stage 1 and buying it back at the end of stage 1, she can likely make a large profit. Thus, we expect to see preying behavior similar to the behavior in the one stage framework discussed in Section II. Our hypothesis is verified by the numerical results shown in Figure 12.

It might be surprising that the asset position  $X_i(T_1)$  of the predators at the end of the first stage changes from a short position to a long position as the number of predators increases. This can be explained in the following way. For a small number of predators the price evolution will be sufficiently close to the one shown in Figure 11, therefore preying is attractive and the predators will enter stage 2 with a short position. As the number of predators increases, the price curve flattens within the first stage due to the increased competition for profit from predatory trading<sup>27</sup> (Figure 13). In comparison, the recovery of prices between stage 1 and stage 2 now becomes attractive, even though it is relatively small. Similar to the line of argument in elastic markets, it now pays off for the predators to acquire a small asset position during stage 1 in order to sell it during stage 2. This is illustrated in Figure 14. If the number of predators is small, it is beneficial to enter stage 2 with a short position; if the number of predators is large, it is more attractive to enter stage 2 with a long position.

Based on this line of argument, we expect the price overshooting to disappear if the number of predators is large. A single predator however can decrease or increase price overshooting, depending on how plastic the market is. In the plastic market considered in this section, even a single predator reduces price overshooting; if the permanent impact is increased to 7.0 and all other parameters are unchanged, a single predator increases price overshooting.

Similar to the results of Section II, we might be tempted to expect that the return for the seller is decreasing as the number of predators increases and predation becomes more fierce. Figure 15 shows that this is not the case. The return for the seller is significantly decreased by predators; furthermore, two predators decrease it more than a single predator. However, the return for the seller is higher when three predators are active than when only two predators are active; as soon as at least two predators are active, each additional predator is beneficial for the seller.

The connection between the return for the seller and the number of predators is a combination of

<sup>27</sup>See also Proposition A.1 in the appendix.



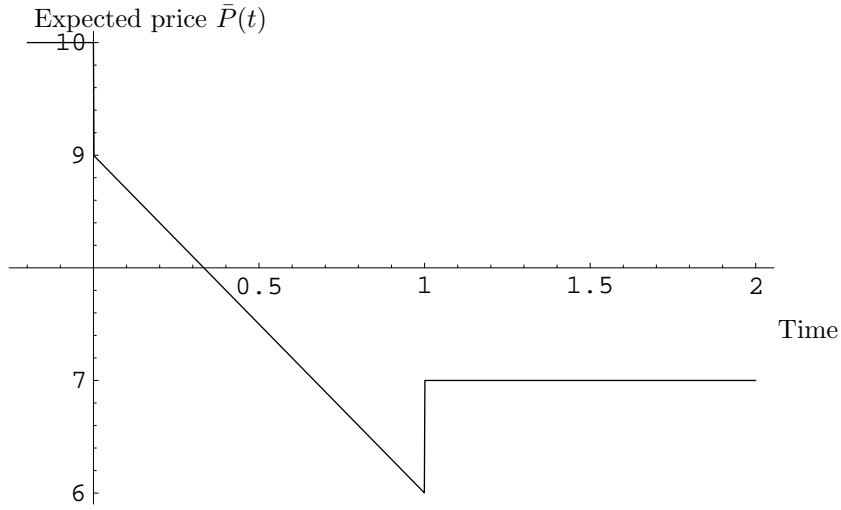


Figure 11: Expected price  $\bar{P}(t)$  in a plastic market over time when no predators are active; at time  $t = 1$ , stage 1 ends and stage 2 begins.

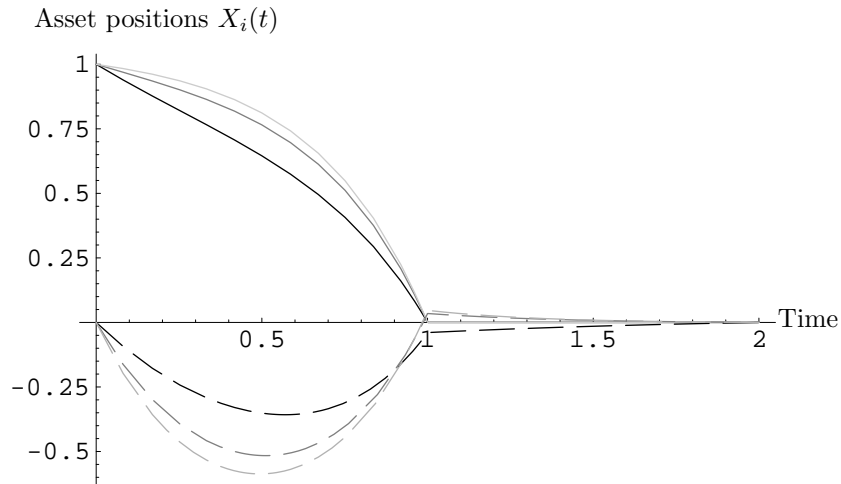


Figure 12: Asset positions  $X_i(t)$  over time in a plastic market; at time  $t = 1$ , stage 1 ends and stage 2 begins. The solid lines represents the seller, the dashed lines the combined asset position of all  $n$  predators. The black lines correspond to  $n = 2$ , the dark grey lines to  $n = 10$  and the light grey lines to  $n = 100$ .

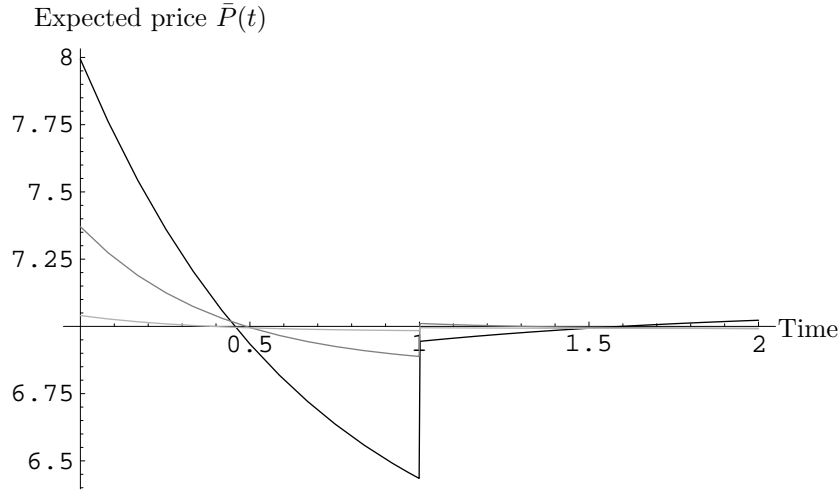


Figure 13: Expected price  $\bar{P}(t)$  in a plastic market over time depending on the number of predators  $n$ ; at time  $t = 1$ , stage 1 ends and stage 2 begins. The black line corresponds to  $n = 2$ , the dark grey line to  $n = 10$  and the light grey line to  $n = 100$ . A significant reduction in price drift can be observed.

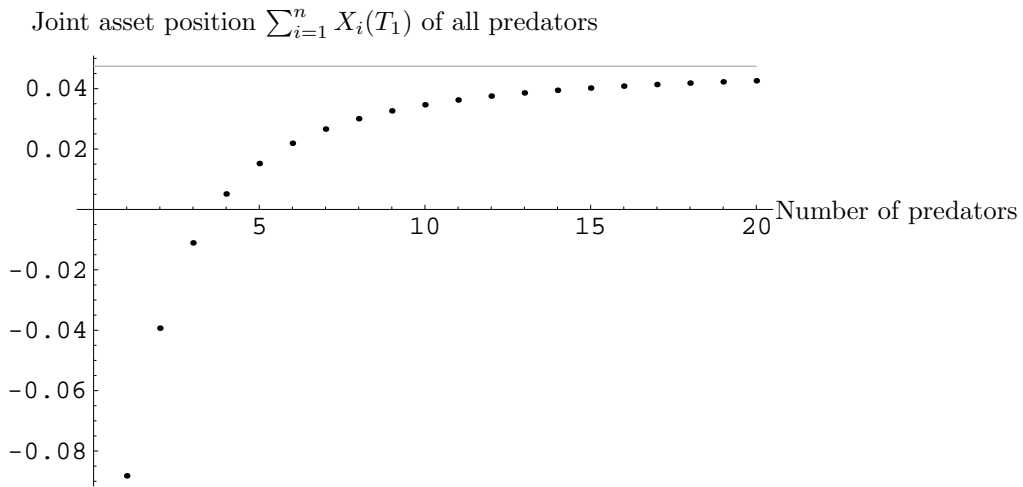


Figure 14: Joint asset position  $\sum_{i=1}^n X_i(T_1)$  of all predators in a plastic market at time  $T_1$  depending on the total number  $n$  of all predators. The grey line represents the limit  $\lim_{n \rightarrow \infty} \sum_{i=1}^n X_i(T_1)$ .

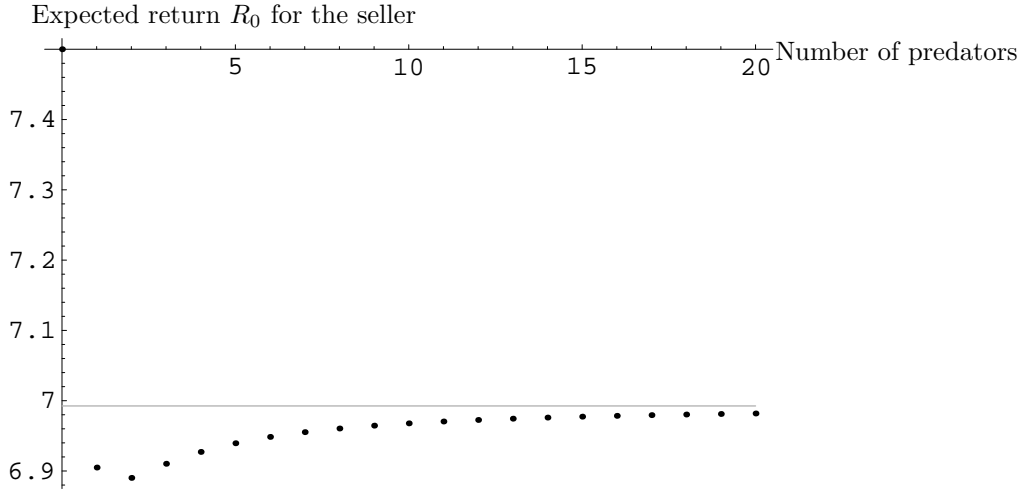


Figure 15: Expected return  $R_0$  for the seller in a plastic market, depending on the number of predators. The grey line represents the limit  $n \rightarrow \infty$ . The return for the seller without predators is at the intersection of  $x$ - and  $y$ -axis.

effects from the one stage model and the two stage model in an elastic market. The first effect (already observed in the one stage model) is that a larger number of predators leads to more aggressive preying and hence to a reduced return for the seller. This effect is very strong for a small number of predators, but not for a large number of predators. The second effect is that a larger number of predators also results in an increased total asset position  $\sum_{i=1}^n X_i(T_1)$  of all predators at the end of stage 1. This reduces the trading pressure in stage 1 and therefore increases the return for the seller. The latter effect dominates the first if the number of predators is large.

### D Example market 3: Intermediate market

In most cases, the differences between the temporary and permanent impact factors  $\gamma$  and  $\lambda$  will not be as extreme as depicted above. If the two parameters are closer together, we can expect to observe characteristics of both elastic as well as plastic markets:

- At the beginning of the first stage, the predators “race the seller to market”, that is, they sell in parallel to her. We say that *intra-stage predation* occurs.
- For a small number of predators, the predators end the first stage with either a long or a short position depending on whether the market is more elastic or more plastic (see Figure 16).
- For a large number of predators, the predators buy back more shares than they sold at the beginning of stage 1; we say that *inter-stage cooperation* takes place subsequently to the intra-stage predation.
- If the number of predators is large, then prices do not overshoot. Instead, market prices are almost flat and almost the same in stage 1 and stage 2.
- If a certain minimum number of predators is active, then additional predators increase the return for the seller since the increase in inter-stage cooperation outweighs the increase in intra-stage predation.

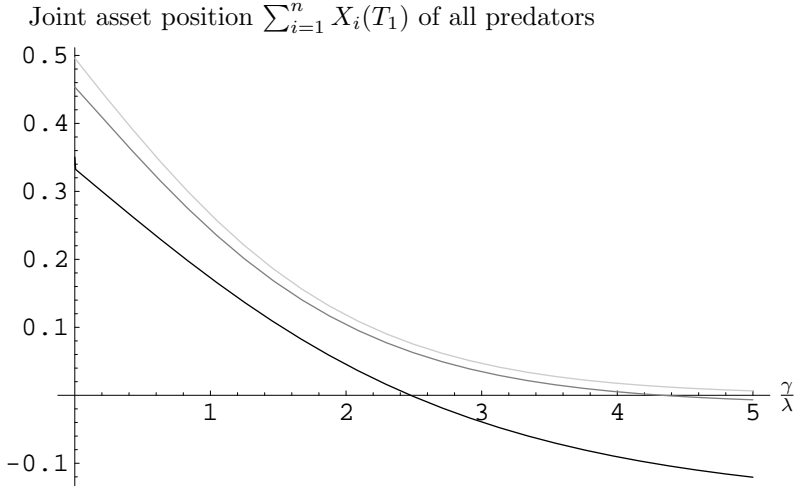


Figure 16: Asset position  $X_1(T_1)$  of the predators, depending on  $\frac{\gamma}{\lambda}$ . The black line corresponds to  $n = 2$ , the dark grey line to  $n = 10$  and the light grey line to  $n = 100$ . The other parameters are chosen as in Table II.

All of these characteristics hold; we prove them in general in the next section. However, one interesting question remains open so far. We have already seen that in elastic markets the seller benefits from predators, whereas in plastic markets the seller prefers to have no predators at all. What is the situation in an intermediate market? Of course, both effects may apply depending on whether the market is more plastic or more elastic in nature. However, a new phenomenon can also arise: It might be the case that a small number of predators is harmful to the seller's profits, but a large number increases the profits even beyond the case of no predation (see Figure 17 for an example).

The practical implications are evident: If there are already some informed traders or if the seller expects to be able to attract a sufficient number of predators, announcing her trading intentions can be attractive; if there is only a limited number of potential predators she is best advised to conceal her intentions.

## VI General properties of the two stage model

After having reviewed three explicit market examples, we summarize their common equilibrium properties.

### A Competitor behavior: Predatory trading versus liquidity provision

**Proposition 3.** *As the number of predators  $n$  tends to infinity, the combined asset position of all predators at the end of stage 1 converges to*

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n X_i(T_1) = \lim_{n \rightarrow \infty} nX_1(T_1) = \frac{e^{\frac{\gamma(T_2 - T_1)}{\lambda}} - 1}{e^{\frac{\gamma(T_2)}{\lambda}} - 1} X_0. \quad (15)$$

In economic terms, this implies that for large  $n$ , intra-stage cooperation between the seller and the predators occurs regardless of the market parameters: in stage 1, the predators buy a portion of the seller's asset position and sell this portion in stage 2. Thereby the market impact in stage 1 is reduced.

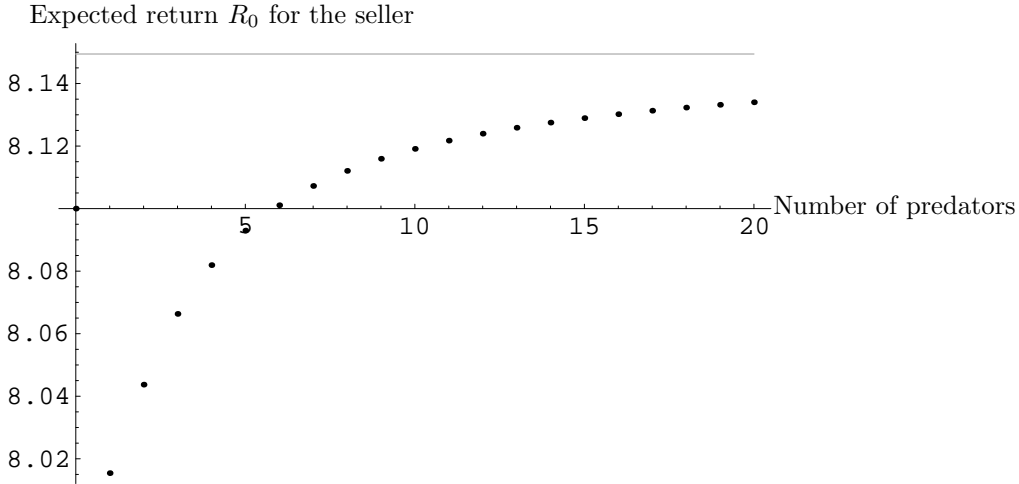


Figure 17: Expected return  $R_0$  for the seller in an intermediate market, depending on the number of predators. The grey line represents the limit  $n \rightarrow \infty$ . The return for the seller without predators is at the intersection of  $x$ - and  $y$ -axis.

We can draw an intuitive consequence for elastic markets: If the number of predators is high, then the net sale of seller and predators in each stage is proportional to the time available for selling. The following corollary expresses this in mathematical terms when sending  $\lambda$  to  $\infty$ .

**Corollary 4.** *As the number of predators  $n$  and the temporary price impact coefficient  $\lambda$  tend to infinity, the combined asset position of all predators at time  $T_1$  converges:*

$$\lim_{\lambda \rightarrow \infty} \lim_{n \rightarrow \infty} \sum_{i=1}^n X_i(T_1) = \frac{T_2 - T_1}{T_2} X_0 \quad (16)$$

We summarize the drivers of inter-stage cooperation.

**Corollary 5.** *For a large enough number  $n$  of predators, the total net amount of liquidity  $\sum_{i=1}^n X_i(T_1)$  provided by strategic players in stage 1 is*

- *decreasing in  $\gamma T_1 / \lambda$ ,*
- *increasing in  $T_2 / T_1$ , and*
- *increasing in  $n$ .*

The first driver highlights the importance of the market environment; inter-stage cooperation is reduced in plastic markets<sup>28</sup>. The second driver relates to the influence of risk management. If the competitors have enough capital, they will be willing to hold inventory for a long period of time, i.e.,  $T_2 > T_1$ . On the other hand, if they are in a financially weak condition, risk management is likely to limit the maximum holding period  $T_2$  in order to reduce the associated risk. The third driver reflects the effect of limited competition among strategic players. By a combination of the latter two drivers, liquidity can disappear in a self-exciting vicious circle: Financial distress of some market participants can result in a general tightening of risk management practices and a smaller number of players engaging in strategic trading, leading to increased predatory trading and more distressed players.

<sup>28</sup>In the repeated game model of Carlin, Lobo, and Viswanathan (2007), the opposite result is obtained and cooperation is increased in plastic markets.

## B Seller behavior: Stealth versus sunshine trading

We now turn to the return that the seller can expect to receive in a market with a certain number  $n$  of strategic competitors.

**Theorem 6.** *By selling an asset position  $X_0$  in stage 1, the seller receives an average total cash position of*

$$R_0 = X_0 \left( P_0 - \gamma X_0 G \left( \frac{\gamma T_1}{\lambda}, \frac{T_2}{T_1}, n \right) \right). \quad (17)$$

The function  $G$  is given in closed form in the proof in Appendix B. For large  $n$ , the seller's return is

- decreasing in  $\gamma T_1/\lambda$ ,
- increasing in  $T_2/T_1$ , and
- increasing in  $n$ .

It converges to:

$$\lim_{n \rightarrow \infty} R_0 = X_0 \left( P_0 - \gamma X_0 \frac{1}{1 - e^{-\frac{\gamma}{\lambda} T_2}} \right) \quad (18)$$

The cash received in the limit case  $n \rightarrow \infty$  is exactly the initial asset position multiplied by the limit of the expected market price derived in Proposition 8.

Given the result above, the benefits of sunshine trading can easily be quantified<sup>29</sup>. If the seller's intentions remain secret<sup>30</sup>, she can expect a return of<sup>31</sup>

$$X_0 (P_0 - \gamma X_0/2 - \lambda X_0/T_1). \quad (19)$$

Alternatively, she can pre-announce her intentions, attract a large number of predators and thus expect a return of

$$X_0 \left( P_0 - \gamma X_0 \frac{1}{1 - e^{-\frac{\gamma}{\lambda} T_2}} \right). \quad (20)$$

**Corollary 7.** *Assuming that pre-announcement attracts a large number of predators ( $n \approx \infty$ ), sunshine trading is superior to stealth trading if*

$$\frac{1}{2} + \frac{\lambda}{\gamma T_1} > \frac{1}{1 - e^{-\frac{\gamma}{\lambda} T_2}}. \quad (21)$$

*If the predators do not face any material time constraint ( $T_2 \rightarrow \infty$ ), sunshine trading is beneficial if*

$$\frac{\lambda}{\gamma} > \frac{T_1}{2}. \quad (22)$$

In our model, the ratio  $\gamma/\lambda$  of the market liquidity parameters  $\gamma$  and  $\lambda$  and the length of the two stages  $T_1$  and  $T_2 - T_1$  determine whether sunshine trading is beneficial. These drivers are not relevant in existing models. Most notably, sunshine trading is always beneficial in the model used by Admati and Pfleiderer (1991), while it is never beneficial in equilibrium in the model of Brunnermeier and Pedersen (2005).

<sup>29</sup>We assume that pre-announcing a trade does not change market-wide liquidity. In case sunshine traders are structurally special, this can be modeled by changing  $\lambda$  and  $\gamma$  for sunshine trades. For example, Admati and Pfleiderer (1991) assume that sunshine traders are uninformed; their trades should therefore result in a smaller (or possibly even no) permanent price change. This can be incorporated by assuming a smaller value for  $\gamma$  for sunshine trades.

<sup>30</sup>Even without pre-announcement, the market will try to infer the complete trading intentions from the trading pattern observable in the market. Barclay and Warner (1993) and Chakravarty (2001) find that the market reacts strongest to orders of medium size because such orders are most likely to be part of the execution of a large, informed transaction. Such observations should be taken into account when performing "stealth execution".

<sup>31</sup>See Almgren and Chriss (2001) and Almgren (2003) for a discussion of the case without predators.

## C Price evolution

We now analyze the market prices resulting from the combined trading activities of the seller and the predators in more detail. In Figures 9 and 13, we observe that when trading commences in  $t = 0$ , the expected price jumps downward from its level  $\bar{P}(0-) = P_0$  to  $\bar{P}(0)$  due to the temporary impact of the selling. After the initial price jump, the expected price  $\bar{P}(t)$  is exhibiting a downward trend. This indicates that our model market does not fulfill the strong form of the efficient markets hypothesis as introduced by Fama (1970): if relevant information is shared by only a *small* number of market participants, then this information is only slowly reflected in market prices. On the other hand, empirical evidence suggests that capital markets are efficient in the semi-strong sense. We would therefore expect that if the seller's intentions are known by a *sufficiently large* number of market participants, this information is instantaneously fully reflected in market prices. Public information can thus not be used to predict price changes. The following proposition states that this is indeed the case in our market model.

**Proposition 8.** *The absolute value of the drift  $|\dot{\bar{P}}(t)|$  is a decreasing function of  $n$ . In the limit, the expected market price instantaneously jumps to*

$$P_0 - \frac{\gamma}{1 - e^{-\frac{\gamma(T_2)}{\lambda}}} X_0 \quad (23)$$

and is constant from thereon throughout stage 1 and stage 2 until the end of stage 2.

In plastic markets, the initial price jump  $|\bar{P}(0) - P_0|$  is an increasing function of the number  $n$  of predators, while it is a decreasing function of  $n$  in elastic markets. It is interesting to note that the new equilibrium price  $P_0 - \frac{\gamma}{1 - e^{-\frac{\gamma(T_2)}{\lambda}}} X_0$  does not depend on whether the seller can trade in stage 2 (see Proposition A.1).

To formally discuss price overshooting, we include the time after  $T_2$  in our analysis, i.e., the time after the seller and the predators have stopped trading. The temporary impact of the trades during  $[0, T_2]$  vanishes immediately at  $T_2$ ; therefore, only the permanent impact remains. The seller sold  $X_0$  while the predators did not change their asset positions. Therefore we obtain an expected market price of  $\bar{P}(T_2+) = P_0 - \gamma X_0$  for the time after  $T_2$ . If during the trading phase  $[0, T_2]$  the price drops below  $\bar{P}(T_2+)$ , i.e.,

$$\min_{t \in [0, T_2]} \bar{P}(t) - \bar{P}(T_2+) < 0 \quad (24)$$

we say that the price *overshoots*. We can now describe the relationship between price overshooting and predatory activity.

**Proposition 9.** *The price  $\bar{P}(t)$  attains its minimum in the interval  $[0, T_2]$  at the end of the first stage:*

$$\min_{t \in [0, T_2]} \bar{P}(t) = \bar{P}(T_1-) \quad (25)$$

*Price overshooting occurs irrespective of the presence of predators:*

$$\bar{P}(T_1-) < \bar{P}(T_2+). \quad (26)$$

*The level of price overshooting  $\bar{P}(T_2+) - \bar{P}(T_1-)$  is increased by predators only in very plastic markets, i.e., only if the permanent impact is much larger than the temporary impact. In all other circumstances, price overshooting is reduced by predators. If predators are already active in the market ( $n \geq 1$ ), then additional predators reduce price overshooting irrespective of the market character.*

It is interesting to compare our results to the models introduced by Brunnermeier and Pedersen (2005) and by Carlin, Lobo, and Viswanathan (2007). Preying introduces price overshooting in the first framework, but it reduces price overshooting in the latter (see Proposition A.2); in our model, the effect of preying on price overshooting depends on the market. In all three models, price overshooting is reduced by additional predators (assuming that at least one predator is active).

## VII Summary and Conclusions

In a number of practical cases, investors need to liquidate large asset positions in a short time. In this paper, we describe optimal liquidation strategies in case other market participants are aware of the investor's needs. A crucial assumption is that these predators are not limited by the same time constraint the seller is facing.

We solve a competitive trading game in an illiquid market model incorporating a temporary and a permanent price impact. Each player faces a dynamic programming problem. According to our model, the optimal strategies for these predators depend on the liquidity characteristics of the market. If the permanent impact affects market prices more heavily than the temporary impact, the predators will "race" the seller to market, selling in parallel with her and buying back after the seller sold her asset position. If price impact is predominantly temporary, predators provide liquidity to the seller by buying some of her shares and selling them after the seller has finished her sale. In the first case, the seller should conceal her trading intentions in order not to attract predators, while in the latter case, pre-announcing a trade can attract liquidity suppliers and thus be beneficial.

As a special case, we investigate behavior in a market with a very large number of predators. We find that in spite of illiquidity, such a market efficiently determines a new price. Information about the seller's intentions is immediately incorporated into the market price and does not affect it thereafter. The predators might race the seller to market, but even in markets with high permanent impact, they quickly start buying back shares and sell these after the seller has finished her sale.

In conclusion, we believe that our model enhances the understanding of stealth and sunshine trading as well as liquidity provision and predation in the marketplace.

## A Propositions on the one stage model

We first state two propositions concerning the one stage model introduced in Section II. These are used for comparison of the one stage model and the two stage model as well as in the proofs presented in Appendix B.

**Proposition A.1.** *In the one stage model, the absolute value of the drift  $|\dot{P}(t)|$  is a decreasing function of  $n$ . In the limit case  $n \rightarrow \infty$ , the expected market price instantaneously jumps to*

$$P_0 - \frac{\gamma}{1 - e^{-\frac{\gamma T_1}{\lambda}}} X_0 \quad (27)$$

and is constant from thereon until the end of trading at time  $t = T_1$ .

**Proof of Proposition A.1.** Using the notation from Theorem 1, the combined trading speed of the seller and all predators amounts to

$$\sum_{i=0}^n \dot{X}_i(t) = \sum_{i=0}^n (ae^{-\frac{n}{n+2} \frac{\gamma}{\lambda} t} + b_i e^{\frac{\gamma}{\lambda} t}) = (n+1)ae^{-\frac{n}{n+2} \frac{\gamma}{\lambda} t}. \quad (28)$$

The change in combined asset position at time  $t$  is therefore:

$$\sum_{i=0}^n (X_i(t) - X_i(0)) = \sum_{i=0}^n \int_0^t \dot{X}_i(s) ds = \int_0^t \sum_{i=0}^n \dot{X}_i(s) ds \quad (29)$$

$$= \int_0^t (n+1)ae^{-\frac{n}{n+2} \frac{\gamma}{\lambda} s} ds = (n+1) \frac{n+2}{n} \frac{\lambda}{\gamma} a \left(1 - e^{-\frac{n}{n+2} \frac{\gamma}{\lambda} t}\right) \quad (30)$$



Now, we can compute the expected market price:

$$\bar{P}(t) = P_0 + \gamma \sum_{i=0}^n (X_i(t) - X_i(0)) + \lambda \sum_{i=0}^n \dot{X}_i(t) \quad (31)$$

$$= P_0 + \gamma(n+1) \frac{n+2}{n} \frac{\lambda}{\gamma} a (1 - e^{-\frac{n}{n+2} \frac{\gamma}{\lambda} t}) + \lambda(n+1) a e^{-\frac{n}{n+2} \frac{\gamma}{\lambda} t} \quad (32)$$

$$= P_0 + \lambda \frac{n+1}{n} (n+2 - 2e^{-\frac{n}{n+2} \frac{\gamma}{\lambda} t}) a \quad (33)$$

$$= P_0 + \lambda \frac{n+1}{n} (n+2 - 2e^{-\frac{n}{n+2} \frac{\gamma}{\lambda} t}) \frac{n}{n+2} \frac{\gamma}{\lambda} \left(1 - e^{-\frac{n}{n+2} \frac{\gamma}{\lambda} T_1}\right)^{-1} \frac{-X_0}{n+1} \quad (34)$$

$$= P_0 - \gamma X_0 \frac{1}{1 - e^{-\frac{n}{n+2} \frac{\gamma}{\lambda} T_1}} + \gamma X_0 \frac{2}{n+2} \frac{e^{-\frac{n}{n+2} \frac{\gamma}{\lambda} t}}{1 - e^{-\frac{n}{n+2} \frac{\gamma}{\lambda} T_1}} \quad (35)$$

Only the last term in the expression above is time dependent; its influence decreases with increasing  $n$ . In the limit, we obtain that the expected market price  $\bar{P}(t)$  is constant:

$$\lim_{n \rightarrow \infty} \bar{P}(t) \equiv P_0 - \gamma X_0 \frac{1}{1 - e^{-\frac{\gamma}{\lambda} T_1}} \quad (36)$$

□

**Proposition A.2.** *Without any predators (i.e., nobody is aware of the seller's intentions), the price overshoots by  $\lambda X_0/T_1$ . If predators are present, the price overshooting is reduced to*

$$\frac{n}{n+2} \gamma X_0 \frac{e^{-\frac{n}{n+2} \frac{\gamma}{\lambda} T_1}}{1 - e^{-\frac{n}{n+2} \frac{\gamma}{\lambda} T_1}}, \quad (37)$$

which is a decreasing function of the number  $n$  of predators.

**Proof of Proposition A.2.** Without any predators, the optimal strategy for the seller is to liquidate her asset position linearly:  $X_0(t) = (T_1 - t)X_0/T_1$ . The market price thus drops to

$$\bar{P}(T_1-) = P_0 - \gamma X_0 - \lambda X_0/T_1. \quad (38)$$

and price overshooting amounts to  $\lambda X_0/T_1$ .

From Equation 33, we know the structure of  $\bar{P}(t)$  when predators are present and deduce that the market price decreases to

$$\bar{P}(T_1) = P_0 - \gamma X_0 \frac{1}{1 - e^{-\frac{n}{n+2} \frac{\gamma}{\lambda} T_1}} + \gamma X_0 \frac{2}{n+2} \frac{e^{-\frac{n}{n+2} \frac{\gamma}{\lambda} T_1}}{1 - e^{-\frac{n}{n+2} \frac{\gamma}{\lambda} T_1}}. \quad (39)$$

Thus, the price overshoots with magnitude

$$\bar{P}(T_1) - \bar{P}(T_1-) = \frac{n}{n+2} \gamma X_0 \frac{e^{-\frac{n}{n+2} \frac{\gamma}{\lambda} T_1}}{1 - e^{-\frac{n}{n+2} \frac{\gamma}{\lambda} T_1}}. \quad (40)$$

The monotonicity follows directly. □

## B Proofs for propositions on the two stage model

The proofs of the theorems, propositions and corollaries presented in this paper are given in order of appearance in the main body of text. In order to keep the proofs compact, they sometimes use results that are independently proven later in this appendix.

**Proof of Theorem 2.** The actual computations are lengthy; we will therefore only sketch the approach (more details are available from the authors on request).

Let us first discuss the case  $n = 1$ , i.e., the seller is facing only one predator. By computations similar to the ones in Proposition A.1, we can express the expected market price  $\bar{P}(t)$  as a *linear* function of the seller's asset position  $X_0$  and the predators asset position  $X_1(T_1) = Z_1$  at the end of stage 1. Furthermore, by Theorem 1 the predator's trading speed  $X_1(t)$  is linear in  $X_0$  and  $Z_1$ . Therefore we can then calculate the return for the predator in the two stages as *quadratic* functions of  $X_0$  and  $Z_1$ :

$$Return_{Predator} = Return_{Stage1}(X_0, Z_1) + Return_{Stage2}(X_0, Z_1) \quad (41)$$

Now, we can determine the optimal  $Z_1$  by maximizing the quadratic function  $Return_{Predator}$ , i.e., by determining the root of its derivative, which is a linear function in  $X_0$ .

Let us turn to the case  $n \geq 2$ , i.e., the seller is facing at least two predators. We assume that  $n - 1$  predators acquire optimal asset positions  $X_i(T_1) = Y_i$  for  $1 \leq i \leq n - 1$  and solve for the optimal asset position  $X_n(T_1) = Z_n$  for the last predator. Similar to the case  $n = 1$  discussed above, we can calculate the return for the last predator as a *quadratic* function of  $X_0 + \sum_{i=1}^{n-1} Y_i$  and  $Z_n$ :

$$Return_{Predator_n} = Return_{Stage1}(X_0 + \sum_{i=1}^{n-1} Y_i, Z_n) + Return_{Stage2}(X_0 + \sum_{i=1}^{n-1} Y_i, Z_n) \quad (42)$$

We can again determine the optimal  $Z_n$  by maximizing  $Return_{Predator_n}$  and obtain a *linear* function of  $X_0 + \sum_{i=1}^{n-1} Y_i$ :

$$Z_n^{optimal} = f(X_0 + \sum_{i=1}^{n-1} Y_i) \quad (43)$$

Similarly we obtain the linear equations

$$Z_j^{optimal} = f(X_0 + \sum_{i=1, i \neq j}^n Y_i) \quad (44)$$

for all  $1 \leq j \leq n$ . Since we assumed that  $(Y_1, \dots, Y_n)$  was optimal in the first place, we know that the optimal  $Z_j^{optimal}$  has to be equal to  $Y_j$ ; we therefore obtain

$$Y_j = f(X_0 + \sum_{i=1, i \neq j}^n Y_i) \quad (45)$$

for all  $1 \leq j \leq n$ . The set of linear equations (45) constitutes a symmetric, non-singular linear problem of  $n$  equations in  $n$  variables. Its unique solution therefore has to fulfill  $Y_1 = \dots = Y_n$  and these  $Y_i$  are a linear function of  $X_0$ . By computing this linear function precisely, we obtain the functional form

$$F\left(\frac{\gamma T_1}{\lambda}, \frac{T_2}{T_1}, n\right) = -\frac{A_2 n^2 + A_1 n + A_0}{B_3 n^3 + B_2 n^2 + B_1 n + B_0} \quad (46)$$

with parameters

$$\begin{aligned}
A_0 = & 2 \left( -e^{\frac{\gamma(-T_1+(2+n)T_2)}{(1+n)\lambda}} - e^{\frac{\gamma(n(3+2n)T_1+(2+n)T_2)}{(2+3n+n^2)\lambda}} + \right. \\
& e^{\frac{\gamma((2+2n+n^2)T_1+n(2+n)T_2)}{(2+3n+n^2)\lambda}} + e^{\frac{\gamma((-2+n^2)T_1+(2+n)^2T_2)}{(2+3n+n^2)\lambda}} + \\
& e^{\frac{\gamma(-nT_1+(1+2n)T_2)}{(1+n)\lambda}} - e^{\frac{\gamma(-nT_1+(2+5n+2n^2)T_2)}{(2+3n+n^2)\lambda}} + e^{\frac{n\gamma T_1+\gamma T_2}{\lambda+n\lambda}} - \\
& \left. e^{\frac{\gamma T_1+n\gamma T_2}{\lambda+n\lambda}} \right)
\end{aligned}$$

$$\begin{aligned}
A_1 = & 3e^{\frac{(2+n)\gamma(-T_1+T_2)}{(1+n)\lambda}} - 3e^{\frac{(1+2n)\gamma(-T_1+T_2)}{(1+n)\lambda}} - 3e^{\frac{\gamma(-T_1+T_2)}{\lambda+n\lambda}} + \\
& 3e^{\frac{n\gamma(-T_1+T_2)}{\lambda+n\lambda}} - 2e^{\frac{\gamma(-T_1+(2+n)T_2)}{(1+n)\lambda}} - e^{\frac{n\gamma(-T_1+(2+n)T_2)}{(2+3n+n^2)\lambda}} + \\
& e^{\frac{\gamma((-2+n^2)T_1+(2+n)T_2)}{(2+3n+n^2)\lambda}} - e^{\frac{\gamma(-(4+3n)T_1+(2+n)^2T_2)}{(2+3n+n^2)\lambda}} + \\
& 2e^{\frac{\gamma(-nT_1+(1+2n)T_2)}{(1+n)\lambda}} + e^{\frac{\gamma(-(2+4n+n^2)T_1+(2+5n+2n^2)T_2)}{(2+3n+n^2)\lambda}} + \\
& 2e^{\frac{n\gamma T_1+\gamma T_2}{\lambda+n\lambda}} - 2e^{\frac{\gamma T_1+n\gamma T_2}{\lambda+n\lambda}}
\end{aligned}$$

$$\begin{aligned}
A_2 = & e^{\frac{(2+n)\gamma(-T_1+T_2)}{(1+n)\lambda}} - e^{\frac{(1+2n)\gamma(-T_1+T_2)}{(1+n)\lambda}} - e^{\frac{\gamma(-T_1+T_2)}{\lambda+n\lambda}} + \\
& e^{\frac{n\gamma(-T_1+T_2)}{\lambda+n\lambda}} - e^{\frac{n\gamma(-T_1+(2+n)T_2)}{(2+3n+n^2)\lambda}} + e^{\frac{\gamma((-2+n^2)T_1+(2+n)T_2)}{(2+3n+n^2)\lambda}} - \\
& e^{\frac{\gamma(-(4+3n)T_1+(2+n)^2T_2)}{(2+3n+n^2)\lambda}} + e^{\frac{\gamma(-(2+4n+n^2)T_1+(2+5n+2n^2)T_2)}{(2+3n+n^2)\lambda}}
\end{aligned}$$

and

$$\begin{aligned}
B_0 = & -2 \left( 2e^{\frac{(1+2n)\gamma(-T_1+T_2)}{(1+n)\lambda}} - e^{\frac{\gamma(-T_1+T_2)}{\lambda+n\lambda}} - e^{\frac{n\gamma(-T_1+T_2)}{\lambda+n\lambda}} - \right. \\
& e^{\frac{\gamma(-T_1+(2+n)T_2)}{(1+n)\lambda}} + e^{\frac{n\gamma(-T_1+(2+n)T_2)}{(1+n)(2+n)\lambda}} - 2e^{\frac{\gamma(n(3+2n)T_1+(2+n)T_2)}{(1+n)(2+n)\lambda}} + \\
& e^{\frac{\gamma((-2+n^2)T_1+(2+n)T_2)}{(1+n)(2+n)\lambda}} + e^{\frac{\gamma((-2+n^2)T_1+(2+n)^2T_2)}{(1+n)(2+n)\lambda}} - \\
& e^{\frac{\gamma(-nT_1+(1+2n)T_2)}{(1+n)\lambda}} + e^{\frac{\gamma(-nT_1+(2+5n+2n^2)T_2)}{(1+n)(2+n)\lambda}} - \\
& \left. 2e^{\frac{\gamma(-(2+4n+n^2)T_1+(2+5n+2n^2)T_2)}{(1+n)(2+n)\lambda}} + 2e^{\frac{n\gamma T_1+\gamma T_2}{\lambda+n\lambda}} \right)
\end{aligned}$$

$$\begin{aligned}
B_1 = & 2e^{\frac{(2+n)\gamma(-T_1+T_2)}{(1+n)\lambda}} - e^{\frac{\gamma(-T_1+T_2)}{\lambda+n\lambda}} - e^{\frac{n\gamma(-T_1+T_2)}{\lambda+n\lambda}} - e^{\frac{\gamma(-T_1+(2+n)T_2)}{(1+n)\lambda}} + \\
& e^{\frac{n\gamma(-T_1+(2+n)T_2)}{(1+n)(2+n)\lambda}} + e^{\frac{\gamma\left(\left(-2+n^2\right)T_1+(2+n)T_2\right)}{(1+n)(2+n)\lambda}} - \\
& 2e^{\frac{\gamma\left(\left(2+2n+n^2\right)T_1+n(2+n)T_2\right)}{(1+n)(2+n)\lambda}} - 2e^{\frac{\gamma\left(-\left(4+3n\right)T_1+(2+n)^2T_2\right)}{(1+n)(2+n)\lambda}} + \\
& e^{\frac{\gamma\left(\left(-2+n^2\right)T_1+(2+n)^2T_2\right)}{(1+n)(2+n)\lambda}} - e^{\frac{\gamma(-nT_1+(1+2n)T_2)}{(1+n)\lambda}} + e^{\frac{\gamma\left(-nT_1+\left(2+5n+2n^2\right)T_2\right)}{(1+n)(2+n)\lambda}} + \\
& 2e^{\frac{\gamma T_1+n\gamma T_2}{\lambda+n\lambda}}
\end{aligned}$$

$$\begin{aligned}
B_2 = & 2\left( e^{\frac{(2+n)\gamma(-T_1+T_2)}{(1+n)\lambda}} - 2e^{\frac{\gamma(-T_1+T_2)}{\lambda+n\lambda}} + e^{\frac{n\gamma(-T_1+T_2)}{\lambda+n\lambda}} + \right. \\
& e^{\frac{\gamma\left(\left(-2+n^2\right)T_1+(2+n)T_2\right)}{(1+n)(2+n)\lambda}} - e^{\frac{\gamma\left(\left(2+2n+n^2\right)T_1+n(2+n)T_2\right)}{(1+n)(2+n)\lambda}} - \\
& e^{\frac{\gamma\left(\left(-2+n^2\right)T_1+(2+n)^2T_2\right)}{(1+n)(2+n)\lambda}} - e^{\frac{\gamma(-nT_1+(1+2n)T_2)}{(1+n)\lambda}} + \\
& 2e^{\frac{\gamma\left(-nT_1+\left(2+5n+2n^2\right)T_2\right)}{(1+n)(2+n)\lambda}} - e^{\frac{\gamma\left(-\left(2+4n+n^2\right)T_1+\left(2+5n+2n^2\right)T_2\right)}{(1+n)(2+n)\lambda}} + \\
& \left. e^{\frac{n\gamma T_1+\gamma T_2}{\lambda+n\lambda}} \right)
\end{aligned}$$

$$\begin{aligned}
B_3 = & -e^{\frac{\gamma(-T_1+T_2)}{\lambda+n\lambda}} + e^{\frac{n\gamma(-T_1+T_2)}{\lambda+n\lambda}} + e^{\frac{\gamma(-T_1+(2+n)T_2)}{(1+n)\lambda}} - e^{\frac{n\gamma(-T_1+(2+n)T_2)}{(1+n)(2+n)\lambda}} + \\
& e^{\frac{\gamma\left(\left(-2+n^2\right)T_1+(2+n)T_2\right)}{(1+n)(2+n)\lambda}} - e^{\frac{\gamma\left(\left(-2+n^2\right)T_1+(2+n)^2T_2\right)}{(1+n)(2+n)\lambda}} - e^{\frac{\gamma(-nT_1+(1+2n)T_2)}{(1+n)\lambda}} + \\
& e^{\frac{\gamma\left(-nT_1+\left(2+5n+2n^2\right)T_2\right)}{(1+n)(2+n)\lambda}}
\end{aligned}$$

Note that the denominator  $B_3n^3 + B_2n^2 + B_1n + B_0$  of the general expression

$$X_i(T_1) = -\frac{A_2n^2 + A_1n + A_0}{B_3n^3 + B_2n^2 + B_1n + B_0}X_0 \quad (47)$$

is 0 in the case  $n = 1$ ; however, the general expression as a whole converges for  $n \rightarrow 1$  against the optimal value of  $X_1(T_1)$  for  $n = 1$  as given in Equation 12.

In the following proofs, we will need the limits  $\lim_{n \rightarrow \infty} A_i$  and  $\lim_{n \rightarrow \infty} B_i$ . All of these limits exist and can be established by direct calculations. We obtain:

$$\lim_{n \rightarrow \infty} A_0 = 2e^{\frac{\gamma T_1}{\lambda}} \left(-1 + e^{\frac{\gamma T_1}{\lambda}}\right) \left(-1 + e^{\frac{\gamma(T_2 - T_1)}{\lambda}}\right)^2 \quad (48)$$

$$\lim_{n \rightarrow \infty} A_1 = -3 \left(-1 + e^{\frac{\gamma T_1}{\lambda}}\right) \left(-1 + e^{\frac{\gamma(T_2 - T_1)}{\lambda}}\right)^2 \quad (49)$$

$$\lim_{n \rightarrow \infty} A_2 = - \left(-1 + e^{\frac{\gamma T_1}{\lambda}}\right) \left(-1 + e^{\frac{\gamma(T_2 - T_1)}{\lambda}}\right)^2 \quad (50)$$

$$\lim_{n \rightarrow \infty} B_0 = -2 \left(-1 + e^{\frac{\gamma T_1}{\lambda}}\right) \left(-1 + e^{\frac{\gamma(T_2 - T_1)}{\lambda}}\right) \left(-1 + 2e^{\frac{\gamma T_1}{\lambda}} - 2e^{\frac{\gamma(T_2 - T_1)}{\lambda}} + e^{\frac{\gamma T_2}{\lambda}}\right) \quad (51)$$

$$\lim_{n \rightarrow \infty} B_1 = \left(-1 + e^{\frac{\gamma T_1}{\lambda}}\right) \left(-1 + e^{\frac{\gamma(T_2 - T_1)}{\lambda}}\right) \left(-1 + e^{\frac{\gamma T_2}{\lambda}}\right) \quad (52)$$

$$\lim_{n \rightarrow \infty} B_2 = 4 \left(-1 + e^{\frac{\gamma T_1}{\lambda}}\right) \left(-1 + e^{\frac{\gamma(T_2 - T_1)}{\lambda}}\right) \left(-1 + e^{\frac{\gamma T_2}{\lambda}}\right) \quad (53)$$

$$\lim_{n \rightarrow \infty} B_3 = \left(-1 + e^{\frac{\gamma T_1}{\lambda}}\right) \left(-1 + e^{\frac{\gamma(T_2 - T_1)}{\lambda}}\right) \left(-1 + e^{\frac{\gamma T_2}{\lambda}}\right) \quad (54)$$

$$(55)$$

□

**Proof of Proposition 3.** We apply Theorem 2 and obtain:

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n X_i(T_1) = - \frac{\lim_{n \rightarrow \infty} A_2}{\lim_{n \rightarrow \infty} B_3} X_0 \quad (56)$$

From the proof of Theorem 2, we know the values of the limits of  $A_2$  and  $B_3$  and the desired result follows. □

**Proof of Corollary 4.** Using Proposition 3 and L'Hospitale's rule, we calculate

$$\lim_{\lambda \rightarrow \infty} \lim_{n \rightarrow \infty} \sum_{i=1}^n X_i(T_1) = \lim_{\lambda \rightarrow \infty} \frac{e^{\frac{\gamma(T_2 - T_1)}{\lambda}} - 1}{e^{\frac{\gamma T_2}{\lambda}} - 1} = \frac{T_2 - T_1}{T_2} \quad (57)$$

□

**Proof of Corollary 5.** We observe that by Theorem 2 all derivatives of  $X_i(T_1)$  converge locally uniformly. Hence, we have

$$\lim_{n \rightarrow \infty} \frac{d}{d\gamma} X_i(T_1) = \frac{d}{d\gamma} \lim_{n \rightarrow \infty} X_i(T_1) \quad (58)$$

and by computing the derivatives of  $\lim_{n \rightarrow \infty} X_i(T_1)$  using Proposition 3 we obtain the first two relations of the corollary.

Similar to the proof of Theorem 6, it can be shown that for large  $n$ ,  $X_i(T_1)$  is increasing in  $n$ . This shows the last of the three relations stated in the corollary. □

**Proof of Theorem 6.** Using Theorems 1 and 2 and Propositions A.1 and 8, we can calculate the return for the seller in a straightforward way and obtain:

$$R_0 = X_0 \left( P_0 - \gamma X_0 \frac{A_7 n^7 + A_6 n^6 + A_5 n^5 + A_4 n^4 + A_3 n^3 + A_2 n^2 + A_1 n + A_0}{B_7 n^7 + B_6 n^6 + B_5 n^5 + B_4 n^4 + B_3 n^3 + B_2 n^2 + B_1 n + B_0} \right) \quad (59)$$

$$=: X_0 \left( P_0 - \gamma X_0 G \left( \frac{\gamma T_1}{\lambda}, \frac{T_2}{T_1}, n \right) \right) \quad (60)$$

The coefficients  $A_i$  and  $B_i$  are functions of  $\frac{\gamma T_1}{\lambda}$ ,  $\frac{T_2}{T_1}$  and  $n$ . They are of a similar structure as the coefficients derived in the proof of Theorem 2, but even more complex. The calculations and coefficients are omitted here for brevity (they are available from the authors on request).

The coefficients  $A_i$  and  $B_i$  converge for  $n \rightarrow \infty$ ; furthermore, their derivatives  $\frac{dA_i}{dn}$  and  $\frac{dB_i}{dn}$  converge to 0 as  $n \rightarrow \infty$ . We compute

$$\lim_{n \rightarrow \infty} R_0 = \lim_{n \rightarrow \infty} \mathbb{E}(\text{Return for the seller}) = X_0 \left( P_0 - \gamma X_0 \frac{\lim_{n \rightarrow \infty} A_7}{\lim_{n \rightarrow \infty} B_7} \right). \quad (61)$$

Inserting  $A_7$  and  $B_7$  and computing the limit gives the desired limit.

To prove that  $\lim_{n \rightarrow \infty} R_0$  is increasing for large  $n$ , we compute the derivative of the seller's return  $R_0$  with respect to  $n$  as

$$\frac{d}{dn} R_0 = -\gamma X_0 \frac{\text{Numerator}}{(B_7 n^7 + B_6 n^6 + B_5 n^5 + B_4 n^4 + B_3 n^3 + B_2 n^2 + B_1 n + B_0)^2} \quad (62)$$

with

$$\begin{aligned} \text{Numerator} = & \left( 7A_7 B_7 n + 7A_7 B_6 + 6A_6 B_7 + \frac{dA_7}{dn} B_7 n^2 + \frac{dA_7}{dn} B_6 n \right. \\ & \left. + \frac{dA_7}{dn} B_5 + \frac{dA_6}{dn} B_7 n + \frac{dA_6}{dn} B_6 + \frac{dA_5}{dn} B_7 \right) n^{12} \\ & - \left( 7B_7 A_7 n + 7B_7 A_6 + 6B_6 A_7 + \frac{dB_7}{dn} A_7 n^2 + \frac{dB_7}{dn} A_6 n \right. \\ & \left. + \frac{dB_7}{dn} A_5 + \frac{dB_6}{dn} A_7 n + \frac{dB_6}{dn} A_6 + \frac{dB_5}{dn} A_7 \right) n^{12} + o(n^{11}). \end{aligned} \quad (63)$$

For large  $n$ , we can omit the  $o(n^{11})$  term; furthermore, we know that all derivatives converge to 0 as  $n \rightarrow \infty$ . We therefore obtain for large  $n$ :

$$\begin{aligned} \text{Numerator} \approx & \left( \left( \frac{dA_7}{dn} B_7 - \frac{dB_7}{dn} A_7 \right) n^2 \right. \\ & \left. + \left( \frac{dA_7}{dn} B_6 + \frac{dA_6}{dn} B_7 - \frac{dB_7}{dn} A_6 n - \frac{dB_6}{dn} A_7 \right) n \right. \\ & \left. + A_7 B_6 - B_7 A_6 \right) n^{12} \end{aligned} \quad (64)$$

Inserting the expressions for  $A_i$  and  $B_i$ , we obtain

$$\lim_{n \rightarrow \infty} \left( \frac{dA_7}{dn} B_7 - \frac{dB_7}{dn} A_7 \right) n^2 = 0 \quad (65)$$

$$\lim_{n \rightarrow \infty} \left( \frac{dA_7}{dn} B_6 + \frac{dA_6}{dn} B_7 - \frac{dB_7}{dn} A_6 n - \frac{dB_6}{dn} A_7 \right) n = 0 \quad (66)$$

$$\lim_{n \rightarrow \infty} (A_7 B_6 - B_7 A_6) = -e^{\frac{\gamma T_1}{\lambda}} \left( e^{\frac{\gamma T_1}{\lambda}} - 1 \right)^7 \left( e^{\frac{\gamma(T_2 - T_1)}{\lambda}} - 1 \right)^5 \left( e^{\frac{\gamma T_2}{\lambda}} - 1 \right)^3 \gamma^2 < 0. \quad (67)$$

The derivative of the seller's return has the opposite sign of the *Numerator* and is thus positive for large values of  $n$ .

To prove that the seller's return is decreasing in  $\gamma T_1/\lambda$  and increasing in  $T_2/T_1$  for large  $n$ , we proceed similar to the proof of Corollary 5, observe that the derivatives of  $R_0$  converge locally uniformly for  $n \rightarrow \infty$  and obtain the desired relations by inspection of the limit  $\lim_{n \rightarrow \infty} R_0$ .  $\square$

**Proof of Corollary 7.** The condition

$$\frac{1}{2} + \frac{\lambda}{\gamma T_1} > \frac{1}{1 - e^{-\frac{\gamma}{\lambda} T_2}}. \quad (68)$$

is obtained by direct comparison of the returns of sunshine and stealth trading given in Equations 19 and 20. Equation 22 can be derived by passing to the limit  $T_2 \rightarrow \infty$ .  $\square$

**Proof of Proposition 8.** First, we note that by arguments similar to the proof of Proposition A.1 (in particular Formula (35)), the price during stage 1 ( $t \in [0, T_1]$ ) is

$$\begin{aligned} \bar{P}(t) = & P_0 - \gamma \left( X_0 - \sum_{i=1}^n X_i(T_1) \right) \frac{1}{1 - e^{-\frac{n}{n+2} \frac{\gamma}{\lambda} T_1}} \\ & + \gamma \left( X_0 - \sum_{i=1}^n X_i(T_1) \right) \frac{2}{n+2} \frac{e^{-\frac{n}{n+2} \frac{\gamma}{\lambda} t}}{1 - e^{-\frac{n}{n+2} \frac{\gamma}{\lambda} T_1}} \end{aligned} \quad (69)$$

and the price during stage 2 ( $t \in [T_1, T_2]$ ) is

$$\begin{aligned} \bar{P}(t) = & P_0 - \gamma \left( X_0 - \sum_{i=1}^n X_i(T_1) \right) - \gamma \left( \sum_{i=1}^n X_i(T_1) \right) \frac{1}{1 - e^{-\frac{n-1}{n+1} \frac{\gamma}{\lambda} (T_2 - T_1)}} \\ & + \gamma \left( \sum_{i=1}^n X_i(T_1) \right) \frac{2}{n+1} \frac{e^{-\frac{n-1}{n+1} \frac{\gamma}{\lambda} (t - T_1)}}{1 - e^{-\frac{n-1}{n+1} \frac{\gamma}{\lambda} (T_2 - T_1)}}. \end{aligned} \quad (70)$$

Again, the time-dependent terms vanish as  $n$  increases. For the first stage, we obtain the limit

$$\begin{aligned} \lim_{n \rightarrow \infty} \bar{P}(t) = & P_0 - \gamma \left( X_0 - \lim_{n \rightarrow \infty} \sum_{i=1}^n X_i(T_1) \right) \frac{1}{1 - e^{-\lim_{n \rightarrow \infty} \frac{n}{n+2} \frac{\gamma}{\lambda} T_1}} \\ & + \gamma \left( X_0 - \lim_{n \rightarrow \infty} \sum_{i=1}^n X_i(T_1) \right) \left( \lim_{n \rightarrow \infty} \frac{2}{n+2} \right) \frac{e^{-\lim_{n \rightarrow \infty} \frac{n}{n+2} \frac{\gamma}{\lambda} t}}{1 - e^{-\lim_{n \rightarrow \infty} \frac{n}{n+2} \frac{\gamma}{\lambda} T_1}} \end{aligned} \quad (71)$$

$$= P_0 - \gamma \left( X_0 - \frac{e^{\frac{\gamma(T_2 - T_1)}{\lambda}} - 1}{e^{\frac{\gamma T_2}{\lambda}} - 1} X_0 \right) \frac{1}{1 - e^{-\frac{\gamma}{\lambda} T_1}} \quad (72)$$

$$= P_0 - \gamma X_0 \frac{e^{\frac{\gamma T_2}{\lambda}}}{e^{\frac{\gamma T_2}{\lambda}} - 1} \quad (73)$$

For the second stage, we compute

$$\begin{aligned}
\lim_{n \rightarrow \infty} \bar{P}(t) &= P_0 - \gamma \left( X_0 - \lim_{n \rightarrow \infty} \sum_{i=1}^n X_i(T_1) \right) \\
&\quad - \gamma \left( \lim_{n \rightarrow \infty} \sum_{i=1}^n X_i(T_1) \right) \frac{1}{1 - e^{-\lim_{n \rightarrow \infty} \frac{n-1}{n+1} \frac{\gamma}{\lambda} (T_2 - T_1)}} \\
&\quad + \gamma \left( \lim_{n \rightarrow \infty} \sum_{i=1}^n X_i(T_1) \right) \lim_{n \rightarrow \infty} \frac{2}{n+1} \frac{e^{-\lim_{n \rightarrow \infty} \frac{n-1}{n+1} \frac{\gamma}{\lambda} (t - T_1)}}{1 - e^{-\lim_{n \rightarrow \infty} \frac{n-1}{n+1} \frac{\gamma}{\lambda} (T_2 - T_1)}} \tag{74}
\end{aligned}$$

$$\begin{aligned}
&= P_0 - \gamma \left( X_0 - \frac{e^{\frac{\gamma(T_2 - T_1)}{\lambda}} - 1}{e^{\frac{\gamma T_2}{\lambda}} - 1} X_0 \right) \\
&\quad - \gamma \left( \frac{e^{\frac{\gamma(T_2 - T_1)}{\lambda}} - 1}{e^{\frac{\gamma T_2}{\lambda}} - 1} X_0 \right) \frac{1}{1 - e^{-\frac{\gamma}{\lambda} (T_2 - T_1)}} \tag{75}
\end{aligned}$$

$$= P_0 - \gamma X_0 \frac{e^{\frac{\gamma T_2}{\lambda}}}{e^{\frac{\gamma T_2}{\lambda}} - 1}. \tag{76}$$

□

**Proof of Proposition 9.** By Formulas 69 and 70, it is easy to see that within each stage the price  $\bar{P}(t)$  moves monotonously. Therefore, the only four possible times at which the minimum price can be achieved are  $T_0$ ,  $T_1 -$ ,  $T_1$  and  $T_2$ . It is straightforward to calculate the prices for these four points in time using Theorem 2 and Formulas 69 and 70, to show that  $\bar{P}(T_1 -)$  is the minimum of these four values and that it is lower than  $\bar{P}(T_2 +)$ . Furthermore, it is direct to show that  $\bar{P}(T_1 -)$  is an increasing function of the number of predators  $n$ .

The different effect of predators on price overshooting in plastic and elastic markets is shown by the examples in Section V. □



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